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A CHARACTERIZATION OF FREE ABELIAN GROUPS

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ABSTRACT. In the category of abelian groups, being free is equivalent to having a discrete norm.

If G is an abelian group then ν is said to be a norm on G if:

(1) $\nu: G \to \mathbf{R}$,

(2) $\nu(g) > 0$ if $g \neq 0$,

(3) $\nu(g+h) \leqslant \nu(g) + \nu(h)$,

(4) $\nu(mg) = |m|\nu(g)$ if $m \in \mathbf{Z}$.

The norm ν is discrete if there is some $\rho > 0$ such that $\nu(g) > \rho$ whenever $g \neq 0$. In other words, ν is discrete if it induces the discrete topology on G. It is easy to find a discrete norm on any free abelian group. In fact, J. Lawrence **[6]** and F. Zorzitto **[9]** have independently shown that the *countable* discretely normed abelian groups are precisely the free ones. It will be shown below that the restriction to countable groups is not necessary. This question originated in the calculation of certain cohomology groups in the Ph.D. thesis of W. Ralph **[7]**.

I take this opportunity to express my thanks to Frank Okoh for several illuminating discussions on this topic. Indeed, he points out that the theorem for countable groups is more easily proved by relying on the following well-known theorem: *Any subgroup of* $\mathbf{R}^n$ *which is discrete is also free* **[1**, p. 102]. To prove the theorem for countable groups it suffices to prove it for groups of finite rank. Suppose G is a finite rank abelian group with discrete norm ν. Identify G with $\mathbf{Z}^n$ for some n and extend the norm ν to a norm $\bar{\nu}$ on $\mathbf{Q}^n$ by divisibility. Then extend $\bar{\nu}$ to a norm $\bar{\bar{\nu}}$ on $\mathbf{R}^n$ by taking limits. Then $\bar{\bar{\nu}}$ is a seminorm on $\mathbf{R}^n$. If it is actually a norm then it is equivalent to the Euclidean norm and hence G is a discrete subset of $\mathbf{R}^n$ and it follows that G is free.

But why is $\bar{\bar{\nu}}$ a norm and not just a seminorm? The answer is provided by the following argument of J. Lawrence. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a point in $\mathbf{R}^n$ other than the origin. Let $r > 0$ be such that $\bar{\bar{\nu}}(\mathbf{z}) > r$ for $\mathbf{z} \in G \setminus \{0\}$ and let $\|\mathbf{x}\| = a$. Choose $t > 0$ such that $t < a$ and the Euclidean ball of radius t is contained in the $\bar{\bar{\nu}}$ ball of radius $r/2$. By **[3**, Theorem 201, p. 170] there are integers $q \neq 0$ and p_i, for $i \leqslant n$, such that $|qx_i - p_i| < t/n$ for $i \leqslant n$. Let $\mathbf{p} = (p_1, \ldots, p_n)$. Then $\|q \cdot \mathbf{x} - \mathbf{p}\| \leqslant t$ and

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hence $\bar{\bar{\nu}}(q \cdot \mathbf{x} - \mathbf{p}) \leqslant r/2$. But note that $\|q \cdot \mathbf{x}\| = |q|a \geqslant a > t$ and hence $\mathbf{p} \neq \mathbf{0}$. It follows that $\bar{\bar{\nu}}(\mathbf{p}) = \nu(\mathbf{p}) > r$ and hence $\bar{\bar{\nu}}(q \cdot \mathbf{x}) \geqslant r/2$.

The proof of the following theorem will use some set theory. For definitions and proofs see either [**4** or **5**].

THEOREM. *An abelian group is free if and only if it is discretely normed.*

PROOF. Suppose not, and let κ be the least cardinal such that there is an abelian group G such that $|G| = \kappa$, G is not free and ν is a discrete norm on G. Without loss of generality, the ρ in the definition of a discrete norm can be assumed to be 1. It has already been shown that $\kappa > \aleph_0$. Also, since every subgroup of G of size less than κ is clearly free, it follows from Shelah's compactness of singular cardinals that κ is a regular cardinal [**8**]. Hence there is a sequence of subgroups $\{G_\alpha;\ \alpha \in \kappa\}$ satisfying the following properties [**2**]:

(1) $|G_\alpha| < \kappa$,

(2) $G_\alpha \subseteq G_\beta$ if $\alpha \in \beta$,

(3) if α is a limit ordinal then $\bigcup\{G_\beta : \beta \in \alpha\} = G_\alpha$,

(4) there is a stationary subset $S \subseteq \kappa$ and $\Phi\colon S \to \kappa$ such that $G_{\Phi(\alpha)}/G_\alpha$ is not free whenever $\alpha \in S$.

It may be assumed that the underlying set of G is κ and in this case it is easy to see that $\{a \in \kappa;$ the underlying set of G_α is $\alpha\}$ is closed and unbounded. Also, $\{\alpha \in \kappa;$ G_α is pure$\}$ is closed and unbounded. Hence, by intersecting S with these two closed and unbounded sets, it is possible to add the following property to the list:

(5) if $\alpha \in S$ then G_α is pure and the underlying set of G_α is α.

Now, if $\alpha \in S$ then $G_{\Phi(\alpha)}/G_\alpha$ is torsion-free but not free. Choose $\gamma_\alpha \leqslant \kappa$ and $\{x^\alpha_\eta;$ $\eta \in \gamma_\alpha\}$ such that $\{[x^\alpha_\eta];\ \eta \in \gamma_\alpha\}$ is an independent subset of $G_{\Phi(\alpha)}/G_\alpha$ whose pure closure is $G_{\Phi(\alpha)}/G_\alpha$. Define a norm μ_α on $G_{\Phi(\alpha)}/G_\alpha$ by $\mu_\alpha(\sum q_\eta[x^\alpha_\eta]) = \sum|q_\eta|\nu(x^\alpha_\eta)$ where the coefficients q_η are rationals.

Since $G_{\Phi(\alpha)}/G_\alpha$ is not free and $|G_{\Phi(\alpha)}/G_\alpha| < \kappa$ it follows that μ_α is not a discrete norm. Hence there is some $y_\alpha \in G_{\Phi(\alpha)} \setminus G_\alpha$ such that $\mu_\alpha([y_\alpha]) < \frac{1}{2}$. Then $k_\alpha y_\alpha = \sum m^\alpha_\eta x^\alpha_\eta + e_\alpha$ where $e_\alpha \in G_\alpha$ and $m^\alpha_\eta/k_\alpha = q^\alpha_\eta$ and m^α_η and k_α are integers. The function $\Psi\colon S \to \kappa$ defined by $\Psi(\alpha) = e_\alpha$ is regressive and hence by Fodor's Theorem there is some $e \in G$ and a stationary subset $S' \subseteq S$ such that $\Psi(\alpha) = e$ whenever $\alpha \in S'$.

Now choose α and $\beta \in S'$ such that $\beta > \Phi(\alpha)$ and $k_\alpha = k_\beta = k$. Then $y_\beta \notin G_\beta \supseteq G_{\Phi(\alpha)}$ and hence $y_\alpha - y_\beta \neq 0$. However,

$$|k|\nu(y_\alpha - y_\beta) = \nu(ky_\alpha - ky_\beta) = \nu\Big(\Big(\sum m^\alpha_\eta x^\alpha_\eta + e\Big) - \Big(\sum m^\beta_\eta x^\beta_\eta + e\Big)\Big)$$
$$\leqslant \sum \big|m^\alpha_\eta\big|\nu\big(x^\alpha_\eta\big) + \sum \big|m^\beta_\eta\big|\nu\big(x^\beta_\eta\big).$$

Hence

$$\nu(y_\alpha - y_\beta) \leqslant \sum \big|q^\alpha_\eta\big|\nu\big(x^\alpha_\eta\big) + \sum \big|q^\beta_\eta\big|\nu\big(x^\beta_\eta\big) = \mu_\alpha([y_\alpha]) + \mu_\beta\big([y_\beta]\big) < 1.$$

This is a contradiction.

As has been noted by Lawrence in [**6**] the above Theorem yields two well-known theorems as immediate corollaries.

COROLLARY 1. *Every Specker group is free.*

COROLLARY 2. *If an abelian group has a positive definite Yamabe function then it is free.*

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