PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 3, March 1985

A RING WHICH IS A DOMAIN LOCALLY BUT NOT GLOBALLY

MOSHE ROITMAN

ABSTRACT. We present here a connected commutative ring R which is not a domain, but R_P is a domain for any prime ideal P of R.

It is well known that if R is a connected (= without nontrivial idempotents) commutative noetherian ring such that R_M is a domain for any maximal ideal M of R, then R itself is a domain [**1**, 13.14]. We show here that this assertion does not remain true if we drop the noetherian assumption on R, thus answering the question in [**1**, 13]. We define inductively a sequence of rings R_n $(n \geq 0)$:

Let $R_0 = F[X,Y]/(XY)$, where F is a field of characteristic 2. In case $n > 0$, adjoin to R_{n-1} indeterminates $X_{a,b}$ for any a, b in R_{n-1} such that $ab = 0$. Let I_n be the ideal in the polynomial ring $R_{n-1}[X_{a,b}]_{\{a,b\in R_{n-1}: ab=0\}}$ generated by $\{aX_{a,b}, b(1-X_{a,b}): a,b \in R_{n-1}, ab = 0\}$. Let $R_n = R_{n-1}[X_{a,b}]/I_n$. We have canonical homomorphisms $\varphi_n: R_n \to R_{n+1}$. Let $R = \lim\operatorname{ind} R_n$.

PROPOSITION. *$R \neq 0$ is connected, R has zero divisors, but R_P is a domain for any prime ideal P.*

To prove the Proposition we need some lemmas. We denote T, a commutative ring with $1 \neq 0$; a, b are elements of T such that $ab = 0$; I is the ideal $(aX, b(1-X))$ in $T[X]$; $T' = T[X]/I$ and φ: $T[X] \to T'$ is the natural homomorphism.

LEMMA 1. *We have $t^2 = 0$ for any $t \in T \cap I$.*

PROOF. Let $t \in T \cap I$, $t = faX + gb(1-X)$, where f, g are in $T[X]$. Then $t = g(0)b = f(1)a$, so $t^2 = g(0)f(1)ab = 0$.

LEMMA 2. *If $t \in T$ is not nil in T, then $\varphi(t)$ is not nil in T'.*

PROOF. If $t \in T$ and $\varphi(t)$ is nil in T', $(\varphi(t))^m = 0$, then $t^m \in T \cap \ker\varphi$, so by Lemma 1, t^m is nil, t is nil.

LEMMA 3. *If T is connected,* $\operatorname{char} T = 2$, *then T' is also connected.*

PROOF. Let $e(X) = e_0 + e_1X + \cdots + e_kX^k \in T[X]$, $\varphi(e)$ idempotent in T'. As $(e_0 + e_1X + \cdots + e_kX^k)^2 - (e_0 + \cdots + e_kX^k) \in (aX, b(1-X))$, we obtain in T: $e_i \in \sqrt{(a,b)}$ for $1 \leq i \leq k$. Indeed, assume $(a,b) \neq (1)$, $\overline{T} = T/(a,b)$ and for $t \in T$, let $\bar{t}$ be the canonical image of t in T. Then $\bar{e}_0 + \cdots + \bar{e}_kX^k$ is an idempotent in $\overline{T}[X]$, so $\bar{e}_i$ is nil for $1 \leq i \leq k$, that is $e_i \in \sqrt{(a,b)}$ for $1 \leq i \leq k$.

If r is sufficiently big, then $e_i^{2^r} \in (a,b)$ for $1 \leq i \leq k$, so there exists t in T such that

$$(e_0 + \cdots + e_kX^k)^{2^r} = e_0^{2^r} + \cdots + e_k^{2^r}X^{2^rk} \equiv t \pmod{(aX, b(1-X))}.$$

Received by the editors January 26, 1984 and, in revised form, May 18, 1984.
1980 *Mathematics Subject Classification.* Primary 13G05, 13A17.

It follows $t^2 \equiv t \pmod{I}$, and so by Lemma 1, we have $(t^2 - t)^2 = t^4 - t^2 = 0$. Therefore $t^2 = 0$ or $t^2 = 1$ and so $e^{2^{r+1}} \equiv 0 \pmod{I}$ or $e^{2^{r+1}} \equiv 1 \pmod{I}$. As $e^{2^{r+1}} \equiv e \pmod{I}$, the lemma is proved.

PROOF OF THE PROPOSITION. Let $\psi\colon R_0 \to R$ be the canonical homomorphism. From Lemma 2 it follows that ψ is a monomorphism, so $R \neq 0$ has zero divisors. From Lemma 3 it follows that R is connected. By construction, if $xy = 0$ in R, then $\mathrm{Ann}(x) + \mathrm{Ann}(y) = R$, so R_P is a domain for any prime ideal P of R.

REFERENCES

1. A. V. Geramita and C. Small, *Introduction to homological methods in commutative rings*, Queen's Papers in Pure and Applied Mathematics, No. 43, Queen's Univ., Kingston, Ontario, 1976.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, MOUNT CARMEL, HAIFA 31999, ISRAEL