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ON THE EXPONENT OF NORM RESIDUE GROUPS

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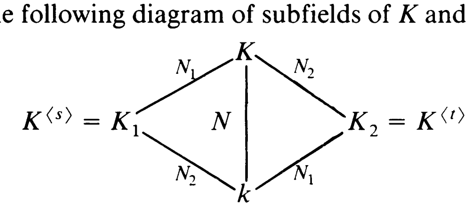
ABSTRACT. We compute the exponent of some norm residue groups in the number theoretic case (global fields). We use the method of Galois cohomology and the theory of the Brauer group over a global field.

Let n be a natural number and let K/k be a Galois extension of arbitrary fields with bicyclic group $G = \mathbf{Z}/n \times \mathbf{Z}/n$ generated by s and t. By $H^r(G, K^*)$, $r \in \mathbf{Z}$, we mean the Tate cohomology groups of G with coefficients in the multiplicative group of K. We consider the following diagram of subfields of K and relative norms:



We use the known structure of the third Galois cohomological groups (see [**4** and **5**]):

$$H^3(G, K^*) = Z^3/B^3 \cong N_1K_2^* \cap N_2K_1^*/NK^*, \tag{1}$$

where the 3-cocycles are

$$Z^3 = \{c = (a, b) \in K_1^* \times K_2^* \mid N_2 a N_1 b = 1\}$$

and the 3-coboundaries are

$$B^3 = \{c = (a, b) \in K_1^* \times K_2^* \mid \text{ there exists } (x, y, z) \in K_1^* \times K_2^* \times K^* \text{ such that } t(x)N_1 z = ax,\ s(y) = byN_2 z\}.$$

The sets Z^3 and B^3 are multiplicative groups through the operation $(a, b)(a', b') = (aa', bb')$. The isomorphism in (1) is induced by the map $Z^3 \to N_1K_2^* \cap N_2K_1^*/NK^*$ which sends $c = (a, b)$ to $N_2 a = N_1 b^{-1} \bmod NK^*$.

The Diophantine equation $N(x) = a^n$, $a \in k^*$, x an indeterminate with value in K, is of interest.

LEMMA. *Let K/k be a Galois extension of arbitrary fields with group $G = \mathbf{Z}/n \times \mathbf{Z}/n$, $n \in \mathbf{N}$. If $H^3(G, K^*) = 0$, then k^*/NK^* has exponent at most n, that is, every nth power of k^* is a norm.*

PROOF. For all $a \in k^*$, the 3-cocycle $c = (a, a^{-1})$ is a coboundary. Hence the equation $N(x) = a^n$ has a solution.

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It is possible to improve on this.

THEOREM 1. *Let K/k be a Galois extension of global fields with group $G = Z/p \times Z/p$, p an odd prime. Then the abelian group k^*/NK^* is of exponent p.*

PROOF. As $k^*/NK^* = H^0(G, K^*)$ has exponent at most p^2, it suffices to show that $a^p = N(x)$ has a k-rational point for every $a \in k^*$. The equation $N(x) = a^p$ can be written as $N_1(aN_2(x^{-1})) = 1$. Using Hilbert 90, it suffices to find $d \in K_2^*$, $x \in K^*$ such that

$$a = N_2(x)s(d)d^{-1}. \tag{2}$$

We interpret this equation in terms of central simple algebras. We introduce the cyclic crossed products $A = (K/K_2, a)$, $B = (K_1/k, a)$ and $C = (K/K_2, d)$. In the Brauer group, we have the following similarity (square brackets denote similarity):

$$[A] = [B \otimes_k K_2] \qquad [\mathbf{3}, (29.13)]. \tag{3}$$

We write C^s for the algebra C with the new K_2-module structure defined by $k \cdot d = s(k)d$, $k \in K_2$, $d \in C$. One can show that $C^s = (K/K_2, s(d))$. We observe that solving (2) is the same as finding a central simple K_2-algebra C such that

$$[A] = \left[C^s \otimes_{K_2} C^{\mathrm{op}}\right]. \tag{4}$$

If $a \in N_2K^*$, there is nothing to show. Hence we suppose that A is not the trivial algebra. We use the local invariants of Hasse for the description of the Brauer group of a global field [**3**, Chapter 8, or **2**, Chapter VII]. We will need the exact sequence [**3**, (32.14)] for the extension K/K_2. As A is obtained from B by extension of the base field, we have from [**2**, Theorem 4, p. 113]

$$(A/w) = n_v(B/v) \quad \text{for all places } w \text{ of } K_2 \text{ above the place } v \text{ of } k. \tag{5}$$

Here $(A/w), (B,/w)$ respectively n_v denote the local invariants of the classes of A, B respectively the local degree of the extension $(K_2)_w/k_v$. Thus the computation of the local invariants for the class of A is reduced to a computation concerning the algebra B. Let w be a place of K_2 and v its restriction to k. Three cases are possible.

Case 1. If w is an infinite place, it is clear that $(A/w) = 0$. Indeed, if $(A/w) = \frac{1}{2}$, then the local index 2 divides the exponent p of the algebra A, which is impossible.

Case 2. If w is a finite place invariant under the action of s, then $n_v = p$ and $(A/w) = (B/v)p$. But $(B/v) = 0$ or s_v/p, with $(s_v, p) = 1$, since $[B] \in \mathrm{Br}(K_1/k)$ is of exponent 1 or p. It follows that $(A/w) = 0$.

Case 3. If the finite place w is not invariant under s, we have p places $w = w_1, w_2, \ldots, w_p$ above v and $n_v = 1$. It follows that $(A/w_i) = (B/v) = 0$ or s_v/p with $(s_v, p) = 1$.

As the next step, we construct a class $[C]$ by giving its local invariants. If $(A/w) = 0$, we put $(C/w) = 0$. We remark that s is transitive on $w_1, w_2, \ldots, w_p$ and that $(C^s/w^s) = (C/w)$. If $(A/w_i) = s_v/p$ for $i = 1, \ldots, p$, we take the sequence of local invariants

$$\{(C/w_1), (C/w_2), \ldots, (C/w_p)\} = \{s_v/p, 2s_v/p, \ldots, (p-1)s_v/p, 0\}$$

such that after appropriate numbering of the w_i's, we have

$$\{(C^s/w_1),\ldots,(C^s/w_p)\} = \{2s_v/p, 3s_v/p,\ldots,0, s_v/p\}.$$

As $\sum_{w|v}(C/w) = (p-1)s_v/2 \equiv 0 \mod \mathbf{Z}$ if p is odd, the class $[C]$ is uniquely determined. The field K splits C, since if $(C/w) \neq 0$, then $(A/w) \neq 0$, and so K has local degree p at w. From the theory of crossed products and the fact that $\mathrm{Br}(K/K_2) \cong H^2(\langle t\rangle, K^*) \cong K_2^*/NK^*$, there exists $d \in K_2^*$ such that $C = (K/K_2, d)$. By construction, the class of $C^s \otimes_{K_2} C^{\mathrm{op}}$ possesses the same local invariants as the class of A. It follows that the algebra $(K/K_2, a^{-1}s(d)d^{-1})$, which is similar to the algebra $A^{\mathrm{op}} \otimes C^s \otimes C^{\mathrm{op}}$, is a matrix algebra. Therefore we have $a^{-1}s(d)d^{-1} \in N_2K^*$ and the proof is complete.

REMARK. The above proof is not valid if we omit the assumption p is odd. In fact there are biquadratic bicyclic extensions K/k such that k^*/NK^* is of exponent 4 as we will show below. This illustrates once more the difference in number theory between 2 and the other primes.

THEOREM 2. *Let K/k be a Galois extension of global fields with group $G = \mathbf{Z}/2 \times \mathbf{Z}/2$. Then we have*:

(I) *If $H^3(G, K^*) = 0$, then k^*/NK^* is of exponent* 2.

(II) *If $H^3(G, K^*) \neq 0$, then k^*/NK^* is of exponent* 4.

PROOF. In view of the lemma, it remains to show (II). The first author has computed that $H^3(G, K^*) = k^*/\prod_{i=1}^3 N_iK_i^*$, where N_3 is the norm from $K_3 = K^{\langle st\rangle}$ to k. (Details of proof will appear elsewhere; a connection with the Hasse problem is given below.) On the other side, we have $\prod_{i=1}^3 N_iK_i^* = \{x \in k^* | x^2 \in NK^*\}$ [**1**, Exercise 5, p. 360]. Since $H^3(G, K^*) \neq 0$, there exists $x \in k^*$ with $x^2 \notin NK^*$. We are done.

REMARK. One can show that $H^3(G, K^*) = N_1K_2^* \cap N_2K_1^*/NK^*$ is equal to the group {local norms}/{global norms}. The question whether a local norm is equal to a global norm is known as the Hasse problem. In the biquadratic case, consider the explicit isomorphism $N_1K_2^* \cap N_2K_1^*/NK^* \cong k^*/\prod_{i=1}^3 N_iK_i^*$, which sends a class $N_1a = N_2b^{-1} \mod NK^*$ to the class $aN_{K/K_1}(d) \mod \prod_{i=1}^3 N_iK_i^*$, where d satisfies $ab = st(d)d^{-1}$. Using this isomorphism and Exercise 5 in [**1**], the first author has produced an algorithm which solves the Hasse problem in this particular case (details will appear elsewhere).

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