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A DOUBLE WEIGHT EXTRAPOLATION THEOREM

C. J. NEUGEBAUER

ABSTRACT. If an operator is of weak type (p_0, p_0) with weights (u, v) for every $(u, v) \in A_{p_0}$, then the same holds for $1 < p < p_0$.

1. A nonnegative function u on $\mathbf{R}^n$ is said to be in A_p iff $(\int_Q u) \cdot (\int_Q u^{1-p'})^{p-1} \leqslant c|Q|^p$, and a pair of nonnegative functions (u, v) is in A_p iff $(\int_Q u)(\int_Q v^{1-p'})^{p-1} \leqslant c|Q|^p$. The smallest c for which these inequalities hold for all cubes $Q \subset \mathbf{R}^n$ will be referred to as the A_p-constant of u, (u, v), respectively. These classes were introduced by Muckenhoupt [3] and are important in the study of weighted norm inequalities for the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|,$$

the Hilbert transform, and many others. In [1] we find the following remarkable theorem.

THEOREM 1. *Let T be a sublinear operator, $1 \leqslant p_0 < \infty$, and*

$$u\{x : |Tf(x)| > y\} \leqslant \frac{B}{y^{p_0}} \int |f|^{p_0} u \quad \text{for every } u \in A_{p_0},$$

where B depends only on the A_{p_0}-constant of u. Then, if $1 < p < \infty$ and $u \in A_p$,

$$u\{x : |Tf(x)| > y\} \leqslant \frac{C}{y^p} \int |f|^p u,$$

where C depends only on the A_p-constant of u.

This theorem remains true if all weak type inequalities are replaced by strong type inequalities, and in this setting was first proved by Rubio de Francia [5]. The method of the proof of Theorem 1 given by Garcia-Cuerva in [1] is different and, as we shall see, can be modified to obtain a double weight (u, v) version of Theorem 1.

2. The results which we obtain are best stated in the terminology of $L(p, q, \lambda)$-spaces [2]. If λ is a Borel measure on $\mathbf{R}^n$ and, for $f: \mathbf{R}^n \to \mathbf{R}$,

$$f_\lambda^*(t) = \inf\{y : \lambda\{|f| > y\} \leqslant t\},$$

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the nonincreasing rearrangement of f with respect to the measure λ, then we write

$$\|f\|_{p,q;\lambda} = \left\{ \int_0^\infty \left[t^{1/p} f_\lambda^*(t) \right]^q \frac{dt}{t} \right\}^{1/q}, \quad \text{if } 1 \leqslant p, q < \infty,$$

and $\|f\|_{p,\infty;\lambda} = \sup_{t>0} t^{1/p} f_\lambda^*(t)$, $1 \leqslant p < \infty$. If $d\lambda = u dx$, we write $\|f\|_{p,q;u} = \|f\|_{p,q;\lambda}$. The weak type inequalities in Theorem 1 can then be written as $\|Tf\|_{p,\infty;u} \leqslant C\|f\|_{p,p;v} \equiv C\|f\|_{p,v}$.

The method in [**1**] carries immediately over to prove the following

THEOREM 2. *Let T be a sublinear operator, $1 \leqslant p_0 < \infty$, and $\|Tf\|_{p_0,\infty;u} \leqslant B\|f\|_{p_0,v}$ for every $(u, v) \in A_{p_0}$, where B depends only on the A_{p_0}-constant of (u, v). Let $1 < p < \infty$, and let $(u, v) \in A_p$ such that $\|Mf\|_{p',v^{1-p'}} \leqslant c\|f\|_{p',u^{1-p'}}$ and $\|Mf\|_{p,u} \leqslant c\|f\|_{p,v}$. Then $\|Tf\|_{p,\infty;u} \leqslant C\|f\|_{p,v}$, where C depends only on the A_p-constant of (u, v).*

The proof is exactly the same as the one given in [**1**] except now $G = \{M(g^{1/t}u)v^{-1}\}^t$. The strong type norm inequalities on the maximal function make Lemmas 1, 2 of [**1**] valid statements, i.e.,

LEMMA 1. *Let $(u, v) \in A_p$ for some $1 < p < \infty$, and for $0 < t \leqslant 1$ and $g \geqslant 0$, let $G = \{M(g^{1/t}u)v^{-1}\}^t$.*

(i) *If $p_0 = p - tp/p'$, then $(gu, Gv) \in A_{p_0}$ with A_{p_0}-constant no larger than the A_p-constant of (u, v) raised to the $1 - t$ power.*

(ii) *If $\|Mf\|_{p',v^{1-p'}} \leqslant C\|f\|_{p',u^{1-p'}}$, then $\|G\|_{p'/t,v} \leqslant C\|g\|_{p'/t,u}$.*

3. It is well known that $(u, v) \in A_p$ if and only if $(v^{1-p'}, u^{1-p'}) \in A_{p'}$, and that these are equivalent with $\|Mf\|_{p,\infty;u} \leqslant B\|f\|_{p,v}$. However, strong type inequalities for M need not hold (see [**4**]), and thus the strong type inequalities for M in the hypothesis of Theorem 2 are unnatural. We will prove the following

THEOREM 3. *Let $1 < p_0 < \infty$, and let T be a sublinear operator such that $\|Tf\|_{p_0,\infty;u} \leqslant B\|f\|_{p_0,v}$ for every $(u, v) \in A_{p_0}$, where B depends only upon the A_{p_0}-constant of (u, v). If $1 < p < p_0$ and $(u, v) \in A_p$, then $\|Tf\|_{p,\infty;u} \leqslant C\|f\|_{p,v}$, where C depends only upon the A_p-constant of (u, v).*

The proof is based on the following

LEMMA 2. *Let $1 < p < p_0 < \infty$, $(u, v) \in A_p$, and $g \geqslant 0$ in $L^{p/(p_0-p)}(v)$. Then there exists a function $G \geqslant 0$ such that*

$$\text{(i)} \qquad u(G > y) \leqslant \frac{c}{y^{p/(p_0-p)}} \int g^{p/(p_0-p)} v, \quad \textit{and}$$

(ii) $(G^{-1}u, g^{-1}v) \in A_{p_0}$, *where the constants involved depend only on the A_p-constant of (u, v).*

PROOF. We note that $p_0' < p'$ and $(v^{1-p'}, u^{1-p'}) \in A_{p'}$. We set

$$t = (p' - p_0')/(p' - 1)$$

so that $p_0' = p' - tp'/p$ and $(p/t)' = p'/p_0'$. We define h by the relation

$$g^{p/(p_0-p)} v = h^{p(p_0-1)/(p_0-p)} v^{1-p'},$$

and we set (as in Lemma 1)

$$H = \left\{ M(h^{1/t} v^{1-p'}) u^{p'-1} \right\}^{t}.$$

Since $(u, v) \in A_p$, we get

$$u\left\{ M(h^{1/t} v^{1-p'}) > y^{1/t} \right\} \leqslant \frac{c}{y^{p/t}} \int h^{p/t} v^{1-p'},$$

and hence

$$u\left\{ H u^{t(1-p')} > y \right\} \leqslant \frac{c}{y^{p/t}} \int h^{p/t} v^{1-p'}. \tag{1}$$

We now let $G = H^{p_0 - 1} u^{-(p_0 - p)/(p-1)}$, and using (1) we check that

$$u\{ G > y \} \leqslant \frac{c}{y^{p/(p_0 - p)}} \int g^{p/(p_0 - p)} v.$$

Since, by Lemma 1, $(hv^{1-p'}, Hu^{1-p'}) \in A_{p_0'}$, we see that

$$\left(H^{1-p_0} u^{(1-p')(1-p_0)}, h^{1-p_0} v^{(1-p')(1-p_0)} \right) \in A_{p_0}.$$

The proof is completed by noting that $G^{-1}u = H^{1-p_0} u^{(p_0-1)(p'-1)}$, and $g^{-1}v = h^{1-p_0} v^{(1-p')(1-p_0)}$.

Proof of Theorem 3. Let $E_s = \{x\colon |Tf(x)| > s\}$. Note that $\|f\|_{p,v}^{p_0} = \| |f|^{p_0} \|_{p/p_0, v} = \int |f|^{p_0} g^{-1} v$ for some $\int g^{p/(p_0-p)} v = 1$. If $r = p/p_0$, then $s^{p_0} u(E_s)^{p_0/p} = s^{p_0} (\int \chi_{E_s} u)^{1/r} = s^{p_0} \{ \int \chi_{E_s} G \cdot G^{-1} u \}^{1/r}$, where G is as in Lemma 2. By Hölder's inequality we obtain $s^{p_0} u(E_s)^{p_0/p} \leqslant s^{p_0} \|\chi_{E_s}\|_{\sigma,1;G^{-1}u}^{1/r} \cdot \|G\|_{\sigma',\infty;G^{-1}u}^{1/r}$, where $\sigma = 1/r$.

We first estimate $\|G\|_{\sigma',\infty;G^{-1}u}$. The distribution function of G with respect to the measure $G^{-1}u\,dx$ is

$$G^{-1}u\{ G(x) > y \} = \int_{\{G(x) > y\}} G^{-1} u \leqslant \frac{1}{y} \int_{\{G(x) > y\}} u \leqslant \frac{c}{y^{p_0/(p_0 - p)}}$$

by (i) of Lemma 2. Hence

$$(G)_{G^{-1}u}^{*}(t) \leqslant \frac{c}{t^{(p_0 - p)/p_0}} = \frac{c}{t^{1/\sigma'}},$$

and so

$$\|G\|_{\sigma',\infty;G^{-1}u} = \sup_{t>0} \left\{ t^{1/\sigma'} (G)_{G^{-1}u}^{*}(t) \right\} \leqslant c.$$

For the estimation of

$$\|\chi_{E_s}\|_{\sigma,1;G^{-1}u} = \int_0^\infty t^{1/\sigma} (\chi_{E_s})_{G^{-1}u}^{*}(t) \frac{dt}{t},$$

we note that

$$\begin{aligned} G^{-1}u\{ \chi_{E_s} > y \} &= G^{-1}u(E_s) \cdot \chi_{[0,1]}(y) \\ &\leqslant \frac{c}{s^{p_0}} \|f\|_{p,v}^{p_0} \cdot \chi_{[0,1]}(y). \end{aligned}$$

From this we obtain that

$$(\chi_{E_s})^*_{G^{-1}u}(t) \leqslant \chi_{[0,R]}(t), \quad \text{where } R = \frac{c}{s^{p_0}} \|f\|^{p_0}_{p,v},$$

and, consequently,

$$\|\chi_{E_s}\|_{\sigma,1;G^{-1}u} \leqslant \int_0^R t^{1/\sigma - 1}\,dt = cR^{1/\sigma} = c\left(\frac{1}{s^{p_0}} \|f\|^{p_0}_{p,v}\right)^r.$$

This implies that

$$s^{p_0} u(E_s)^{p_0/p} \leqslant cs^{p_0} \cdot \frac{1}{s^{p_0}} \|f\|^{p_0}_{p,v} \quad \text{or} \quad su(E_s)^{1/p} \leqslant c\|f\|_{p,v},$$

which was to be proved.

4. It is not known to the writer whether Theorem 3 is true if $p > p_0$. In order to formulate a substitute result for $p > p_0$ we rewrite

$$\int |f|^p v = \int |f|^p v^{p'} v^{1-p'} = \int \left(|f| v^{p'-1}\right)^p v^{1-p'}$$
$$= \int_0^\infty \left(fv^{p'-1}\right)^{*p}_{v^{1-p'}}(t)\,dt = \left\|fv^{p'-1}\right\|^p_{p,p;v^{1-p'}}.$$

THEOREM 4. *Let $1 < p_0 < p < \infty$, and let T be a sublinear operator so that, for every $(u, v) \in A_{p_0}$,*

$$\|Tf\|_{p_0,\infty,u} \leqslant c\|f\|_{p_0 v} = c\left\|fv^{p_0'-1}\right\|_{p_0,p_0;v^{1-p_0'}},$$

where c only depends on the A_{p_0}-constant of (u, v). Then, if $(u, v) \in A_p$,

$$\|Tf\|_{p,\infty,u} \leqslant c\left\|fv^{p'-1}\right\|_{p,p_0;v^{1-p'}},$$

where c only depends upon the A_p-constant of (u, v).

REMARK. Since $\|fv^{p'-1}\|_{p,p_0;v^{1-p'}} \geqslant \|fv^{p'-1}\|_{p,p;v^{1-p'}} = \|f\|_{p,v}$ (see **[2]**), the conclusion of Theorem 4 is not as strong as the corresponding weak type inequality.

The proof is based on a slight reformulation of Lemma 1, which we state as

LEMMA 3. *Let $1 < p_0 < p < \infty$, $(u, v) \in A_p$, and let $t = (p - p_0)/(p - 1)$. If $g \geqslant 0$ is in $L^{p/(p-p_0)}(u)$, and $G = \{M(g^{1/t}u)v^{-1}\}^t$, then*

(i) $$v^{1-p'}\left\{M\left(g^{1/t}u\right)^t > y\right\} \leqslant \frac{c}{y^{p/(p-p_0)}} \int g^{p/(p-p_0)} u, \quad \textit{and}$$

(ii) $(gu, Gv) \in A_{p_0}$.
The constants in (i) *and* (ii) *depend only on the A_p-constant of (u, v).*

PROOF. Since $(v^{1-p'}, u^{1-p'}) \in A_{p'}$ with the same $A_{p'}$-constant as the A_p-constant for (u, v), we see that

$$\int_{\{M(g^{1/t}u) > y^{1/t}\}} v^{1-p'} \leqslant \frac{c}{y^{p'/t}} \|g\|^{p'/t}_{p'/t,u},$$

and this is (i). The proof of (ii) is exactly the same as in **[1]**.

PROOF OF THEOREM 4. Let $E_s = \{x: |Tf(x)| > s\}$ and note that $s^{p_0} u(E_s)^{p_0/p} = s^{p_0}\|\chi_{E_s}\|_{p/p_0,u} = s^{p_0}\int \chi_{E_s} gu$, for some $\|g\|_{(p/p_0)',u} = 1$. By (ii) of Lemma 3,

$$s^{p_0}u(E_s)^{p_0/p} \leqslant c\int |f|^{p_0} Gv = c\int |f|^{p_0} v^{p'-t} M(g^{1/t}u)^t v^{1-p'}$$

$$\leqslant c\left\| |f|^{p_0} v^{p'-t} \right\|_{(p'/t)',1;v^{1-p'}} \left\| M(g^{1/t}u)^t \right\|_{p'/t,\infty;v^{1-p'}}$$

$$\leqslant c\left\| f v^{p'-1} \right\|^{p_0}_{p,p_0;v^{1-p'}},$$

since $(p'/t)' = p/p_0$. This completes the proof.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907