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A CHARACTERIZATION OF SUBSPACES X OF l_p FOR WHICH $K(X)$ IS AN M-IDEAL IN $L(X)$

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ABSTRACT. Given a subspace X of l_p, $1 < p < \infty$, the compact operators on X are an M-ideal in the bounded linear operators on X if and only if X has the compact approximation property.

0. Introduction. Recently Harmand and Lima [7] proved that if X is a Banach space for which $K(X)$, the space of compact operators on X, is an M-ideal in $L(X)$, the space of bounded linear operators on X, then there is a net $\{T_\alpha\}$ in $K(X)$ so that:

(i) $T_\alpha \to I$ strongly,
(ii) $\|T_\alpha\| \leqslant 1$ for all α,
(iii) $T_\alpha^* \to I$ strongly,
(iv) $\lim_\alpha \|I - T_\alpha\| = 1$.

The main result of this paper is a strong converse to the Harmand-Lima theorem for subspaces of l_p, $1 < p < \infty$. In Theorem 6 we show that if X is a subspace of $(\sum X_n)_p$ ($\dim X_n < \infty$; $1 < p < \infty$) which has the compact approximation property, then $K(X)$ is an M-ideal in $L(X)$.

Part of the proof consists in showing that such an X satisfies conditions (i)–(iv) in the Harmand-Lima theorem. This result (which is simple given the state-of-the-art in Banach space theory) is proved for general reflexive spaces in §2.

§3 is devoted to proving the converse of the Harmand-Lima theorem for subspaces of $(\sum X_n)_p$. Here we use blocking methods which have been previously used in the study of isomorphic, rather than isometric, properties of l_p and a few other spaces.

1. Notation and preliminaries. If X and Y are Banach spaces, $L(X, Y)$ (resp. $K(X, Y)$) will denote the space of all bounded linear operators (resp. compact linear operators) from X to Y. If $X = Y$ then we simply write $L(X)$ (resp. $K(X)$). Ball(X) will denote the closed unit ball of X.

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A Banach space X is said to have the compact approximation property (resp., compact metric approximation property) if the identity operator on X is in the closure of $K(X)$ (resp. $\mathrm{Ball}(K(X))$) with respect to the topology of uniform convergence on compact sets in X.

A Banach space X is said to have a finite-dimensional Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ if every $x \in X$ can be uniquely written as $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ and each X_n is a finite-dimensional subspace of X. For each n the partial sum projection P_n on X is defined by $P_n(\sum_{i=1}^{\infty} x_i) = \sum_{i=1}^{n} x_i$, where $x_i \in X_i$. It is easy to see that $\sup_n \|P_n\| < \infty$.

A closed subspace J of a Banach space X is called an L-summand if there is a projection P on X such that $PX = J$ and $\|x\| = \|Px\| + \|(I - P)x\|$ for every $x \in X$. A closed subspace J of X is called an M-ideal if J°, the annihilator of J in X^*, is an L-summand in X^*.

Alfsen and Effros [**1**] and Lima [**8**] characterized M-ideals by intersection properties of balls. We use the following characterization of M-ideals due to Lima [**8**, Theorem 6.17]. A closed subspace J of a Banach space X is an M-ideal of X if and only if for any $\varepsilon > 0$, for any $x \in \mathrm{Ball}(X)$, and for any $y_i \in \mathrm{Ball}(J)$ $(i = 1, 2, 3)$, there exists $y \in J$ such that $\|x + y_i - y\| \leqslant 1 + \varepsilon$ for $i = 1, 2, 3$.

2. Relations among approximation properties. Grothendieck [**5**] proved that if X is a reflexive Banach space or a separable conjugate space which has the approximation property, then X has the metric approximation property. In the case of the compact approximation property, the analogous implication is valid for reflexive Banach spaces.

PROPOSITION 1. *If X is a separable reflexive Banach space which has the compact approximation property, then X has the compact metric approximation property.*

PROOF. The Lindenstrauss-Tzafriri proof [**11**, p. 40] of Grothendieck's theorem proves this. In the notation of that proof one need only observe that for any T in $K(X)$, the function g_T is indeed in $C(K)$.

REMARKS. 1. It is a formal consequence of Proposition 1 that every reflexive space with the compact approximation property also has the compact metric approximation property.

2. We do not know whether Proposition 1 is true if X is only assumed to be a separable conjugate space. To apply the Lindenstrauss-Tzafriri argument one needs to prove that if Y^* is separable, then the weak*-continuous compact operators on Y^* are dense in $K(Y^*)$ when $K(Y^*)$ is given the topology of uniform convergence on compact subsets of Y^*.

COROLLARY 2. *If X is a separable reflexive Banach space which has the compact approximation property, then there is a sequence $\{T_n\}_{n=1}^{\infty}$ in* $\mathrm{Ball}(K(X))$ *so that $T_n \to I_X$ (identity map on X) strongly and $T_n^* \to I_{X^*}$ (identity map on X^*) strongly.*

PROOF. By Proposition 1 there exists a sequence $\{S_n\}_{n=1}^{\infty}$ in $\mathrm{Ball}(K(X))$ so that $S_n \to I$ strongly. Since X is reflexive, $S_n^* x^* \to x^*$ weakly for each $x^* \in X^*$. Since X^* is separable, there are convex combinations T_n of $\{S_i\}_{i=n}^{\infty}$ so that $T_n^* \to I$ strongly.

PROPOSITION 3. *Suppose X is a reflexive subspace of a Banach space Y with the property that there exists a sequence $\{P_n\}_{n=1}^{\infty}$ in $K(Y)$ such that $\overline{\lim}_n \|I_Y - P_n\| \leqslant 1$ and $P_n \to I_Y$ (the identity map on Y) strongly, and suppose X has the compact approximation property. Then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ in* $\mathrm{Ball}(K(X))$ *such that $\overline{\lim}_n \|I_X - T_n\| \leqslant 1$, $T_n \to I_X$ strongly and $T_n^* \to I_{X^*}$ strongly.*

PROOF. Let $\{P_n\}_{n=1}^{\infty}$ be as above, and for each n, let $P_{n|X}\colon X \to Y$ be the restriction of P_n to X. Then $P_{n|X} \to I_X (X \to Y)$ strongly. By Corollary 2 there exists a sequence $\{S_n\}_{n=1}^{\infty}$ in $\mathrm{Ball}(K(X)) \subset \mathrm{Ball}(K(X, Y))$ such that $S_n \to I_X$ strongly and $S_n^* \to I_{X^*}$ strongly. As a sequence of operators from X to Y, we have $P_{n|X} - S_n \to 0$ strongly as $n \to \infty$. Since X is reflexive it follows that $P_{n|X} - S_n \to 0$ weakly in $L(X, Y)$ [**12**, p. 33]. Indeed, the map $S \to x^*(Sy)$ defines an isometry from $K(X, Y)$ to $C(\mathrm{Ball}(X) \times \mathrm{Ball}(Y^*))$, the space of continuous functions on the compact Hausdorff space $\mathrm{Ball}(X) \times \mathrm{Ball}(Y^*)$, where $\mathrm{Ball}(X)$ has the weak topology and $\mathrm{Ball}(Y^*)$ has the weak*-topology. As a sequence in $C(\mathrm{Ball}(X) \times \mathrm{Ball}(Y^*))$, $\{P_{n|X} - S_n\}_{n=1}^{\infty}$ is uniformly bounded and $P_{n|X} - S_n \to 0$ pointwise on $\mathrm{Ball}(X) \times \mathrm{Ball}(Y^*)$. By the Riesz representation theorem and the Hahn-Banach theorem, for any $\phi \in L(X, Y)^*$, there is a regular Borel signed measure μ on $\Omega = \mathrm{Ball}(X) \times \mathrm{Ball}(Y^*)$ such that $\phi(s) = \int_\Omega x^*(Sx)\, d\mu(x, x^*)$ for all $S \in K(X, Y)$. By the bounded convergence theorem, $\phi(P_{n|X} - S_n) \to 0$ as $n \to \infty$.

Since $P_{n|X} - S_n \to 0$ weakly in $L(X, Y)$, there exist sequences $\{Q_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ such that $Q_n = \sum_{k=a_n+1}^{a_{n+1}} \lambda_k P_{k|X}$, $T_n = \sum_{k=a_n+1}^{a_{n+1}} \lambda_k S_k$, and $\|Q_n - T_n\| \to 0$, where $\lambda_k \geqslant 0$, $\sum_{k=a_n+1}^{a_{n+1}} \lambda_k = 1$, and $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers. Obviously $\|T_n\| \leqslant 1$, $\underline{\lim}_n \|I_X - T_n\| \leqslant \overline{\lim}_n \|I_X - Q_n\| \leqslant 1$, $T_n \to I_X$ strongly, and $T_n^* \to I_{X^*}$ strongly.

REMARKS. 1. The relationship between the weak operator topology and the weak topology on the space of operators has been, at least in special cases, known for a long time. The idea of using this relationship to deduce some kind of approximation condition for a subspace from the corresponding condition for the whole space is due to M. Feder [**3**].

2. The analogue of Proposition 3 for nonseparable reflexive spaces can be deduced from Proposition 3 by using Lindenstrauss' decomposition of nonreflexive spaces via transfinite sequences of norm one projections [**10**].

3. M-ideals.

LEMMA 4. *Suppose $\{P_n\}_{n=1}^{\infty}$ is a sequence in $K(Y)$ for a Banach space Y which converges strongly to the identity map on Y and K is a weakly compact subset of Y. Given $\varepsilon > 0$ and a positive integer n, there exists an integer $m = m(n, \varepsilon) > n$ so that*

$$\sup_{y \in K} \min_{n \leqslant k < m} d(P_k y, K) \leqslant \varepsilon,$$

where $d(x, K) = \inf\{\|x - z\| : z \in K\}$ is the distance from x to the set K.

PROOF. If not, there exists a sequence $\{y_m\}_{m=n+1}^{\infty}$ in K so that for each $m = n + 1, n + 2, \ldots$

$$\min_{n \leqslant k < m} d(P_k y_m, K) > \varepsilon.$$

Letting y be any weak cluster point of $\{y_m\}_{m=n+1}^{\infty}$, and using the compactness of the P_k's, we infer that

$$\inf_{n \leqslant k < \infty} d(P_k y, K) \geqslant \varepsilon.$$

This is a contradiction because y is in K and $\|y - P_k y\| \to 0$ as $k \to \infty$.

LEMMA 5. *Let X be a reflexive Banach space which is a subspace of a Banach space Y which has a finite-dimensional Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ with partial sum projections $\{P_n\}_{n=1}^{\infty}$, and set $\alpha = \sup_n\{\|P_n\|\}$. Then for any $\varepsilon > 0$ and $T \in K(X)$ with $\|T\| \leqslant 2$, there exists a positive integer n such that*

(i) $\|(I - P_n)Tx\| \leqslant \varepsilon$ *for every* $x \in \mathrm{Ball}(X)$,

(ii) *if* $x \in \mathrm{Ball}(X)$ *and* $\|P_n x\| \leqslant \varepsilon/4$, *then* $\|Tx\| \leqslant \varepsilon\alpha$.

PROOF. Since the closure of $T(\mathrm{Ball}(X))$ is compact, (i) is true for all large n.

If no n satisfies (ii) then there is a sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathrm{Ball}(X)$ such that $\|P_k x_k\| < \varepsilon/4$ and $\|Tx_k\| > \varepsilon\alpha$. We may assume $x_k \to x \in X$ weakly. We claim that $\|x\| \leqslant \varepsilon\alpha/3$. If not, $\|P_l x\| > \varepsilon\alpha/3$ for all large l. Since $P_l x_k \to P_l x$ in norm as $k \to \infty$, $\|P_l x_k\| \to \|P_l x\| > \varepsilon\alpha/3$. This is impossible, since for $k > l$, $\|P_l x_k\| \leqslant \alpha\|P_k x_k\| < \alpha\varepsilon/4$. Thus, $\|x\| \leqslant \varepsilon\alpha/3$.

Since T is compact and $Tx_k \to Tx$ weakly, $Tx_k \to Tx$ in norm as $k \to \infty$. Thus $\|Tx_k\| \to \|Tx\|$. This is a contradiction because $\|Tx_k\| > \varepsilon\alpha$ for all k and $\|Tx\| \leqslant \|T\|\,\|x\| < 2\varepsilon\alpha/3 < \varepsilon\alpha$.

THEOREM 6. *If X is a closed subspace of $Y = (\sum X_n)_p$ ($\dim X_n < \infty$, $1 < p < \infty$) which has the compact approximation property, then $K(X)$ is an M-ideal in $L(X)$.*

PROOF. Let $S_1, S_2, S_3 \in \mathrm{Ball}(K(X))$ and $T \in \mathrm{Ball}(L(X))$. We show that for any $\eta > 0$ there exists $K \in K(X)$ such that $\|S_i + T - K\| \leqslant 1 + \eta$ $(i = 1, 2, 3)$.

By Proposition 3 we can choose a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathrm{Ball}(K(X))$ so that $\overline{\lim}_n\|I - T_n\| \leqslant 1$, $T_n \to I_X$ strongly, and $T_n^* \to I_{X^*}$ strongly. Fix $1 > \varepsilon > 0$ and choose m so that $\|S_i - T_m S_i\| \leqslant \varepsilon$ for $i = 1, 2, 3$. So for $i = 1, 2, 3$ we have

$$\|S_i + (I - T_n)T\| \leqslant \|T_m S_i + (I - T_n)T\| + \varepsilon \quad \text{for all } n.$$

Let $\{P_n\}$ denote the partial sum projections associated with the natural finite-dimensional decomposition $\{X_n\}_{n=1}^{\infty}$ of Y. Using Lemma 5, with this choice of P_n's (so that $\alpha = 1$), choose M so that for $i = 1, 2, 3$,

(i) if $x \in \mathrm{Ball}(X)$, then $\|(I - P_M)(T_M S_i x)\| \leqslant \varepsilon$,

(ii) if $x \in \mathrm{Ball}(X)$ and $\|P_M x\| \leqslant \varepsilon/4$, then $\|T_m S_i x\| < \varepsilon$.

By Lemma 4 we can choose $N > M$ so that for every $x \in X$, there is $k = k(x)$ $(M \leqslant k < N)$ such that $d(P_k x, X) \leqslant \varepsilon\|x\|$. Given $x \in X$ with $\|x\| = 1$, let $k = k(x)$ and pick $y_1 \in X$ so that $\|P_k x - y_1\| \leqslant \varepsilon$. Setting $y_2 = x - y_1$, we have

(iii) $\|y_2 - (I - P_k)x\| = \|P_k x - y_1\| < \varepsilon$, $\|(I - P_k)y_1\| \leqslant \varepsilon$, and $\|P_k y_2\| \leqslant \varepsilon$.

Finally, choose r large enough so that

(iv) $\|(I - T_r)Ty\| \leqslant 8\varepsilon$ for every y in the set $A = \{y \in X\colon \|y\| \leqslant 2$ and $\|(I - P_n)y\| \leqslant \varepsilon\}$,

(v) $\|P_M(I - T_r)T\| = \|T^*(I - T_r^*)P_M^*\| < \varepsilon$ and $\|I - T_r\| \leqslant 1 + \varepsilon$.

This is possible because A has a 3ε-net and $T_n \to I$ strongly.

For $x \in X$ with $\|x\| = 1$ write $x = y_1 + y_2$ as in (iii). Then for $i = 1, 2, 3$,

$$\begin{aligned}
&\|T_m S_i x + (I - T_r)Tx\|^p \\
&\quad \leqslant \big(\|P_M(T_m S_i x) + (I - P_M)(I - T_r)Tx\| \\
&\qquad + \|(I - P_M)T_m S_i x\| + \|P_M(I - T_r)Tx\|\big)^p \\
&\quad \leqslant \big(\|P_M(T_m S_i x) + (I - P_M)(I - T_r)Tx\| + \varepsilon + \varepsilon\big)^p \quad \text{(by (i) and (v))} \\
&\quad = \|P_M(T_m S_i x)\|^p + \|(I - P_M)(I - T_r)Tx\|^p + f(\varepsilon) \quad (f(\varepsilon) \to 0 \text{ as } \varepsilon \to 0) \\
&\quad \leqslant \big(\|P_M T_m S_i y_1\| + \|P_M T_m S_i y_2\|\big)^p \\
&\qquad + \big(\|(I - P_M)(I - T_r)Ty_1\| + \|(I - P_M)(I - T_r)Ty_2\|\big)^p + f(\varepsilon) \\
&\quad \leqslant \big(\|y_1\| + 8\varepsilon\big)^p + \big(8\varepsilon + (1 + \varepsilon)\|y_2\|\big)^p + f(\varepsilon) \quad \big(\text{by (ii)–(v) since } \|y_1\| \leqslant 2\big) \\
&\quad \leqslant \big(\|P_k x\| + 9\varepsilon\big)^p + \big(\|(I - P_k)x\| + 10\varepsilon\big)^p + f(\varepsilon) \quad \text{(by (iii))} \\
&\quad < \|P_k x\|^p + \|(I - P_k)x\|^p + g(\varepsilon)^p \qquad (g(\varepsilon) \to 0 \text{ as } \varepsilon \to 0) \\
&\quad = 1 + g(\varepsilon)^p.
\end{aligned}$$

Thus for $i = 1, 2, 3$,

$$\|S_i + T - T_r T\| = \|S_i + (I - T_r)T\| \leqslant 1 + \varepsilon + g(\varepsilon).$$

Choose ε so that $\varepsilon + g(\varepsilon) < \eta$ and let $K = T_r T$.

Combining Theorem 6 with the Harmand-Lima theorem, we get the following

COROLLARY 7. *If X is a closed subspace of $(\sum X_n)_p$ (dim $X_n < \infty$), $1 < p < \infty$, then $K(X)$ is an M-ideal in $L(X)$ if and only if X has the compact approximation property.*

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