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MULTIPLIERS FOR EIGENFUNCTION EXPANSIONS OF SOME SCHRÖDINGER OPERATORS

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ABSTRACT. Let G be a graded nilpotent Lie group and let L be a positive Rockland operator on G. Let E_λ denote the spectral resolution of L on $L^2(G)$. A sufficient condition is given under which a function m on $\mathbf{R}^+$ is a L^p-multiplier for L, $1 < p < \infty$; that is $\|\int_0^\infty m(\lambda)\, dE_\lambda f\|_p \leqslant C_p\|f\|_p$ for a constant C_p, $f \in L^p(G) \cap L^2(G)$. Then the same is done for an operator $\pi(L)$, where π is a unitary representation of G induced from a unitary character of a normal connected subgroup H of G. Hence the case of the Hermite operator $-d^2/dx^2 + x^2$ is covered and an L^p-multiplier theorem for classical Hermite expansions is obtained.

1. Let L be an essentially selfadjoint on its domain, positive, densely defined operator on $L^2(X)$, where X is a measure space. Let E_λ be a spectral resolution of the identity for which

$$Lf = \int_0^\infty \lambda\, dE_\lambda f, \qquad f \in \operatorname{Dom}(L).$$

If m is a bounded measurable function on $\mathbf{R}^+$, we write $m(L)$ for the "multiplier operator"

$$m(L)f = \int_0^\infty m(\lambda)\, dE_\lambda f, \qquad f \in L^2(X),$$

which is bounded on $L^2(X)$.

The question about conditions on m which guarantee that $m(L)$ is bounded on $L^p(X)$ has been raised by E. M. Stein [**9**] and discussed in a number of papers afterwards. Thus we say that a function m is an L^p-multiplier for the operator L if there exists a constant C_p such that for all f in $L^p(X) \cap L^2(X)$

$$\|m(L)f\|_p \leqslant C_p\|f\|_p.$$

In [**1**] A. Bonami and J. L. Clerc present a technique, based on investigation of some g-functions of Paley-Littlewood type, which allows them to obtain an L^p-multiplier theorem in the case when L is the Laplace-Beltrami operator on a compact, riemannian manifold. An estimate due to L. Hörmander (cf. [**1**, p. 259]) plays here the crucial role.

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Recently, A. Hulanicki and J. W. Jenkins [**6**, **7**, **8**] have obtained similar estimates for a class of operators which are of different types and include, among others, operators $-\Delta + V$, where Δ is the Laplacian on R^k and V is a sum of squares of polynomials.

In this paper we observe that a method used by Bonami and Clerc can be applied to results of Hulanicki and Jenkins.

We obtain a stronger version of the Hulanicki-Stein multiplier theorem for a sub-Laplacian (cf. [**5**]) and for a positive Rockland operator (cf. [**6**]), and we produce new multiplier theorems for the eigenfunction expansions of Schrödinger type operators.

We illustrate our results by the following theorem concerning the Hermite expansion.

THEOREM 1. *There exists an integer N such that if $m \in C^N(\mathbf{R}^+)$ satisfies the conditions*

$$|m(\lambda)| \leqslant M, \quad \sup_{A>0} \frac{1}{A} \int_0^A \lambda^k |m^{(k)}(\lambda)| d\lambda \leqslant M, \qquad k = 1, \ldots, N,$$

then for every $1 < p < \infty$ the sequence $m(2j+1)$, $j = 0, 1, \ldots,$ is a multiplier on $L^p(\mathbf{R})$ with respect to the Hermite expansion; that is, for all f in $L^p(\mathbf{R}) \cap L^2(\mathbf{R})$

$$\left\| \sum m(2j+1) \langle h_j, f \rangle h_j \right\|_p \leqslant C_p \|f\|_p$$

with a constant C_p.

Here h_j is a jth Hermite function and

$$\langle h_j, f \rangle = \int_{\mathbf{R}} h_j(x) f(x)\, dx.$$

2. Let E_λ be a positive (i.e. $E_\lambda = 0$ for $\lambda \leqslant 0$) spectral resolution of the identity on $L^2(X)$. Denote by S_R^δ, $R > 0$, the δ-Riesz mean

$$S_R^\delta f = \int_0^R \left(1 - \frac{\lambda}{R}\right)^\delta dE_\lambda f, \qquad f \in L^2(X).$$

We suppose that E_λ has the following properties:

(a) For $p > 1$ and a positive integer δ the Marcinkiewicz-Zygmund property holds, i.e.

> there exists a constant B_p such that for an arbitrary sequence of positive numbers R_j and for an arbitrary sequence of functions $f_j \in L^p(X) \cap L^2(X)$ we have

$$\left\| \left(\sum \left| S_{R_j}^\delta f_j \right|^2 \right)^{1/2} \right\|_p \leqslant B_p \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_p. \tag{2.1}$$

(b) The operators $T^t = \int_0^\infty \exp(-\lambda t)\, dE_\lambda$ are contractions on all $L^p(X)$, $1 \leqslant p \leqslant \infty$.

The following theorem is essentially due to Bonami and Clerc (cf. [**1**, p. 260]; the details of the proof extracted from [**1**] and adapted to our situation can be found in [**10**]).

THEOREM A. *Let E_λ be a positive spectral resolution of the identity which satisifies* (a) *and* (b). *If a function $m \in C^{\delta+1}(\mathbf{R}^+)$ satisfies the conditions*

$$|m(\lambda)| \leqslant M, \quad \sup_{A>0} \frac{1}{A}\int_0^A \lambda^j |m^{(j)}(\lambda)| d\lambda \leqslant M, \qquad j = 1,\ldots,\delta+1,$$

then m is an L^p-multiplier for E_λ: that is, there exists a constant C_p such that

$$\left\| \int_0^\infty m(\lambda)\, dE_\lambda f \right\|_p \leqslant C_p \|f\|_p, \qquad f \in L^p(X) \cap L^2(X).$$

The constant C_p does not depend on the function m but only on M.

In the sequel we deal with spaces of homogeneous type, in the sense of [**2**] only.

Let us recall that a topological space X equipped with a continuous pseudometric ρ and with a measure μ, which, for a constant C, satisfies

$$\mu(B_r(y)) \leqslant C\mu\left(B_{r/2}(x)\right), \qquad x, y \in X, r > 0, \tag{2.2}$$

is a space of homogeneous type and the Hardy-Littlewood maximal function

$$m^*f(x) = \sup_{r>0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y)| d\mu(y)$$

is of weak type (1, 1) (cf. [**2**]). Here $B_r(x)$ denotes the ball $B_r(x) = \{y: \rho(x, y) < r\}$.

Moreover, the following Marcinkiewicz-Zygmund type inequality due to C. Fefferman and E. M. Stein is valid (cf. [**4**]; see also [**1**]).

THEOREM B. *Let μ be a measure and ρ a pseudometric on X such that* (2.2) *holds. Then for every $1 < p < \infty$ there exists a constant D_p, such that for an arbitrary sequence of functions f_i we have*

$$\left\| \left(\sum |m^* f_i|^2 \right)^{1/2} \right\|_p \leqslant D_p \left\| \left(\sum |f_i|^2 \right)^{1/2} \right\|_p.$$

Observe that if the spectral measure E_λ on $L^2(X)$ satisfies the estimate

$$\sup_{R>0} \left| S_R^\delta f \right| \leqslant C m^* f, \tag{2.3}$$

then, in virture of Theorem B, E_λ has for every p, $1 < p < \infty$, the Marcinkiewicz-Zygmund property (2.1).

3. Let G be a graded nilpotent Lie group of [**5**]. G endowed with the Haar measure and the pseudometric generated by the homogeneous norm is of homogeneous type (cf. [**5**]).

Let L be a positive Rockland operator on G that is a homogeneous, left-invariant differential operator such that $\pi(L)$ is injective on C^∞-vectors for every irreducible, nontrivial unitary representation π of G. It is known (cf. [**5**]) that L is hypoelliptic, and as such it is essentially selfadjoint on $L^2(G)$.

Let $E_L(\lambda)$ be a spectral resolution of the identity for L. The operators $T^t = \int_0^\infty \exp(-\lambda t)\, dE_L(\lambda)$ are contractions on all $L^p(G)$, $1 \leqslant p \leqslant \infty$ (cf. **[5]**).

As we have mentioned before, Hulanicki and Jenkins have shown (cf. **[6, 8]**; see also [7]) that there exists an integer $N(L)$ such that

$$\sup_{R>0} \left|S_R^{N(L)} f\right| \leqslant C m_G^* f, \tag{3.1}$$

where m_G^* is the Hardy-Littlewood maximal function on G and $S_R^{N(L)}$, $R > 0$, are Riesz means corresponding to $E_L(\lambda)$.

Thus we can state the following

PROPOSITION 1. *Let L be a positive Rockland operator and let $E_L(\lambda)$ be a spectral resolution of L. Let $N(L)$ be an integer such that* (3.1) *holds. If $m \in C^{N(L)+1}(\mathbf{R}^+)$ satisfies the conditions*

$$|m(\lambda)| \leqslant M, \quad \sup_{A>0} \frac{1}{A} \int_0^A \lambda^k |m^{(k)}(\lambda)|\, d\lambda \leqslant M, \qquad k = 1,\ldots,N(L)+1,$$

for a constant M, then for every p, $1 < p < \infty$, the function m is an L^p-multiplier for the operator L.

4. Let G be an arbitrary nilpotent Lie group and let π be a representation of G induced from a unitary character of a normal connected subgroup H of G. The operators $\pi(x)$, $x \in G$, act on $L^2(G/H)$.

Let $X_1,\ldots,X_k$ be elements which generate the Lie algebra of G. Put

$$L = \sum_{j=1}^k (-1)^{n_j} X_j^{2n_j},$$

where n_j, $j = 1,\ldots,k$, are arbitrary positive integers. Then $\pi(L)$ is a positive essentially selfadjoint operator on $L^2(G/H)$. Denote by $E_{\pi(L)}(\lambda)$ the spectral resolution of $\pi(L)$.

PROPOSITION 2. *Let L and $E_{\pi(L)}$ be as above. Then there exists an integer N such that if $m \in C^N(\mathbf{R}^+)$ satisfies the conditions*

$$|m(\lambda)| \leqslant M, \quad \sup_{A>0} \frac{1}{A} \int_0^A \lambda^k |m^{(k)}(\lambda)|\, d\lambda \leqslant M, \qquad k = 1,\ldots,N,$$

for a constant M, then for every p, $1 < p < \infty$, the function m is an L^p-multiplier for the operator $\pi(L)$.

PROOF. Let us repeat, for the reader's convenience, the same arguments as in the proof of Theorem 2.6 of **[8]** which allows us to assume that G is stratified and that L is a Rockland operator. Let $\underline{G}$ be the nilpotent free group of the same nilpotency class as G and let $\underline{X}_1,\ldots,\underline{X}_k$ be the free generators of the Lie algebra of $\underline{G}$. Denote by α the homomorphism of $\underline{G}$ onto G sending $\exp \underline{X}_j$ onto $\exp X_j$. Thus $\pi' = \pi \circ \alpha$ is a representation of $\underline{G}$ induced by a unitary character of the normal connected subgroup $\underline{H} = \alpha^{-1}(H)$ of $\underline{G}$. Define the dilations δ_t, $t > 0$, of free Lie algebras of $\underline{G}$ by putting

$$\delta_t \underline{X}_{-j} = t^{1/2n_j} \underline{X}_j, \qquad 1,\ldots,k.$$

Then

$$\underline{L} = \sum_{j=1}^{k} (-1)^{n_j} \underline{X}_j^{2n_j}$$

is a Rockland operator on $\underline{G}$, $\delta_t \underline{L} = tL$, and $\pi'(\underline{L}) = \pi(L)$.

Proposition 2.2 of **[8]** asserts that G/H, equipped with the Haar measure and the pseudometric ρ defined by

$$\rho(x, y) = \inf\{|xy^{-1}z| : z \in H\},$$

is a space of homogenous type. Denote by $m^*_{G/H}$ the corresponding Hardy-Littlewood maximal function.

Consequently, Theorem 2.6 of **[8]** applied to the function $K(\lambda) = (1 - \lambda)^N$ for $\lambda \leqslant 1$ and $K(\lambda) = 0$ for $\lambda > 1$, for sufficiently large N, allows us to obtain an estimate

$$\sup_{R>0} \left| \int_0^R \left(1 - \frac{\lambda}{R}\right)^N dE_{\pi(L)}(\lambda) f \right| \leqslant C \cdot m^*_{G/H}(f).$$

Now, it remains to note that $\pi(L)$ generates a semigroup of contractions on all $L^p(G/H)$, $1 \leqslant p \leqslant \infty$. Denote by $E_L(\lambda)$ the spectral resolution of L in $L^2(G)$. By **[5]**

$$\int_0^\infty \exp(-\lambda t)\, dE_L(\lambda) f = k_t * f, \qquad f \in L^2(G),$$

where $k_t \in L^1(G)$, $\|k_t\|_1 = 1$. Note that $\{\pi(k_t)\}_{t>0}$ is a semigroup on $L^2(G/H)$ and that, for $f \in L^2(G/H)$ and $\varphi \in C_c(G)$,

$$\lim_{t \to 0} t^{-1}\{\pi(k_t)\pi(\varphi)f - \pi(\varphi)f\} = \pi(L\varphi)f = \pi(L)\pi(\varphi)f.$$

Thus, $\pi(L)$ is the infinitesimal generator for $\{\pi(k_t)\}_{t>0}$, which gives

$$\int_0^\infty \exp(-\lambda t)\, dE_{\pi(L)}(\lambda) f = \pi(k_t) f, \qquad f \in L^2(G/H).$$

The operators $\pi(x)$ act on $L^2(G/H)$ by

$$\pi(x) f(\dot{y}) = a(x, \dot{y}) f(\dot{y}\dot{x}),$$

where the scalar function a is such that $|a(x, y)| = 1$. Thus $\pi(x)$ is a contraction on every $L^p(G/H)$ and so the same is true for the operator

$$\pi(k_t) = \int_G \pi(x) k_t(x)\, dx.$$

This completes the proof of the proposition.

Now, to see that Theorem 1 is a consequence of Proposition 2 we note that the Hermite operator $-d^2/dx^2 + x^2$ is of the form $\pi(L)$, where L is the sub-Laplacian on the Heisenberg group, and π is the Schrödinger representation on it.

Finally we mention also that in virtue of the recent results of W. Cupała [3], following the ideas of Hulanicki and Jenkins [8], our Proposition 2 produces an L^p-multiplier theorem for the eigenfunction expansions of the operators of the form

$$(-1)^k d^{2k}/dx^{2k} + p(x),$$

where $p(x)$ is a positive polynomial and k is an arbitrary positive integer.

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