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CONFORMAL MAPPINGS OF DOMAINS SATISFYING A WEDGE CONDITION

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ABSTRACT. A plane Jordan curve Γ satisfies an interior (exterior) wedge condition if for some $\alpha \in (0,1)$ there is a fixed wedge of opening $\alpha\pi$ such that for any $\omega \in \Gamma$ one may place a wedge inside (outside) Γ with vertex at ω. Let f be a conformal mapping of the disk D onto the interior of Γ. We establish Hölder continuity of $f(f^{-1})$ on $\partial D(\Gamma)$ with best possible exponents in terms of α.

1. Introduction. Let Γ be a closed Jordan curve in the complex ω-plane with interior Ω and exterior Ω^*. We shall say that Γ satisfies an interior α-wedge condition if there exist $r > 0$ and $\alpha \in (0,1)$ such that, for every $\omega \in \Gamma$, a closed circular sector of radius r and opening $\alpha\pi$ lies in $\overline{\Omega}$, with vertex at ω. We say that Γ satisfies an exterior α-wedge condition if, for each $\omega \in \Gamma$ such a wedge lies in $\overline{\Omega}^*$ with vertex at ω. The interior wedge condition is often encountered in the study of partial differential equations where, together with its higher-dimensional analogs, it is called a "cone condition" [**1**, p. 233].

Let f be a conformal mapping of $D = \{\zeta\colon |\zeta| < 1\}$ onto the interior of Γ. Our purpose here is to deduce Hölder continuity of f or f^{-1} on $\overline{D}$ or on $\overline{\Omega}$, respectively, from the appropriate α-wedge condition. The interior α-wedge condition is a special case of the one-sided smoothness condition studied by Pommerenke in [**4**] where, among other results, Hölder continuity of f on $\overline{D}$ is established. The Hölder exponent obtained in [**4**] is not sharp in terms of α, however. We shall obtain the best exponents for both f and f^{-1}.

THEOREM 1. *Suppose that Γ satisfies an interior α-wedge condition. Then f is Hölder continuous on $\overline{D}$ with exponent α.*

THEOREM 2. *Suppose that Γ satisfies an exterior α-wedge condition. Then f^{-1} is Hölder continuous on $\overline{\Omega}$ with exponent $1/(2-\alpha)$.*

That the exponents are best possible is seen by taking Γ to be a polygon.

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2. Geometric lemmas and reduction to strips. In this section we shall derive some purely geometric properties of curves satisfying an α-wedge condition. For two points $\omega_1, \omega_2 \in \Gamma$ we define $C(\omega_1, \omega_2)$ to be the arc of Γ of smaller diameter between ω_1 and ω_2. The interior distance between ω_1 and ω_2 on Γ, relative to Ω, is $d_\Omega(\omega_1, \omega_2) = \inf_\gamma \operatorname{diam} \gamma$, where γ runs over all arcs from ω_1 to ω_2 which lie in Ω, except for their endpoints.

LEMMA 1 (POMMERENKE [**4**, THEOREM 1]). *Suppose that Γ satisfies an interior α-wedge condition. Then there exists a constant $M_1 > 0$, depending only on Γ, such that for $\omega_1, \omega_2 \in \Gamma$*

$$\operatorname{diam} C(\omega_1, \omega_2) \leqslant M_1 d_\Omega(\omega_1, \omega_2).$$

For $\omega_0 \in \Gamma$ and $\rho > 0$, each component of $\{\omega\colon |\omega - \omega_0| = \rho\} \cap \Omega$ is a crosscut of Ω. If $\rho < r$, then exactly one of these crosscuts intersects the α-wedge at ω_0. We shall call this the ρ crosscut of Ω at ω_0.

LEMMA 2. *Suppose that Γ satisfies an interior α-wedge condition and that $0 \in \Gamma$. Let $0, \omega', \omega^*$ be consecutive points on $C(0, \omega^*)$ with $\rho = |\omega^*| < r$, and ω^* on the ρ crosscut of Ω at 0. Then there exists a constant $M_2 > 0$, depending on Γ such that*

$$\left|\frac{\omega'}{\omega^*}\right| < M_2.$$

PROOF. By Lemma 1,

$$|\omega'| \leqslant \operatorname{diam} C(0, \omega^*) \leqslant M_1 d_\Omega(0, \omega^*) < M_1(|\omega^*| + 2\pi|\omega^*|) = M_2|\omega^*|,$$

since there is a path γ from 0 to ω^* in Ω which consists of a ray inside the α wedge at 0 and a subarc of the ρ crosscut at 0.

LEMMA 3. *Let Γ satisfy an interior α-wedge condition and let $0, \omega_1, \omega^*, \omega_2$ be consecutive points on Γ with $\omega^* \in C(\omega_1, \omega_2)$, $|\omega_1| = |\omega_2| = \rho$, and such that ω_1 and ω_2 are endpoints of an arc of the circle $\{|\omega| = \rho\}$ lying in Ω. Then there exists a constant $M_3 > 0$ depending on Γ for which*

$$|\omega^*|/\rho < M_3.$$

PROOF. Using Lemma 1 we have for $|\omega^*| > \rho$,

$$\begin{aligned} |\omega^*| - |\omega_1| \leqslant |\omega^* - \omega_1| &\leqslant \operatorname{diam} C(\omega_1, \omega_2) \\ &\leqslant M_1 d_\Omega(\omega_1, \omega_2) < 2\pi\rho M_1. \end{aligned}$$

Thus $|\omega^*|/\rho < 2\pi M_1 + 1 = M_3$.

Of course, Lemma 3 (and the others) apply with the words "exterior" and "Ω^*" replacing "interior" and "Ω". In this form it will be used to prove Lemma 4, which assumes the exterior condition. It is convenient to state and prove Lemma 4 in the context to which it will be applied in the proof of Theorem 2, so we shall now transform Ω and Ω^* into strip domains.

Suppose that 0 and ω_0 are on Γ and that $|\omega_0| = (\operatorname{diam} \Gamma)/2 = R > 2r$. Suppose also that r is so small that $|\omega_1 - \omega_2| < 2r$ implies that $\operatorname{diam} C(\omega_1, \omega_2) < R/2$. For a particular $\omega' \in \Gamma$, with $|\omega'| < r$, let Γ_1 be the open arc of Γ between 0 and ω_0 which contains ω' and let $\Gamma_2 = \Gamma - \bar{\Gamma}_1$. Next consider the mapping

$$w(\omega) = \log((\omega - \omega_0)/\omega)$$

which may be defined in Ω and Ω^* so that

(1) $w(\Omega)$ is an infinite strip S in the w-plane, bounded by curves $C_1 = w(\Gamma_1)$ and $C_2 = w(\Gamma_2)$ with $-\infty$ and $+\infty$ as boundary points;

(2) $w(\Omega^*)$ is a strip S^* bounded by curves C_1 and C_2', where we may assume that $C_2' = \{w - 2\pi i, w \in C_2\}$.

Now, let L be a piecewise analytic arc from $-\infty$ to $+\infty$ in S, and for u real, let $\Lambda_u = \{w\colon \operatorname{Re} w = u\}$. We define $\sigma(u)$ to be the maximal closed subarc of $\Lambda_u \cap \bar{S}$ which is the first (moving along L from $-\infty$ to $+\infty$) to be crossed by L an odd number of times.

LEMMA 4. *Suppose that Γ satisfies an exterior α-wedge condition and that S and S^* are strip domains corresponding to Ω and Ω^*, as above. Suppose that $w^* = u^* + iv^* \in \sigma(u^*) \cap C_1$ and that w^* separates $w' = u' + iv' \in C_1$ from $+\infty$, with*

$$u' > \log((R - r)/r).$$

Then there exists a constant $M_4' > 0$ depending only on Γ such that

$$u' - u^* < M_4'. \tag{2.1}$$

REMARK. In the ω-plane, this means that for ω^* on the circular crosscut of Ω at 0 corresponding to $\sigma(u^*)$, with $\omega^* \in C(0, \omega')$ and $|\omega'| < r$, we have

$$\left|\frac{\omega^*}{\omega'}\right| < M_4, \quad \text{where } M_4 = \frac{R + r}{R - (R/2)} e^{M_4'}. \tag{2.2}$$

PROOF. We first observe that C_1 and C_2' are "below" L in the sense that they are in the component of $\mathbf{C} - L$ with $-i\infty$ as a boundary point. Similarly C_2 is "above" L.

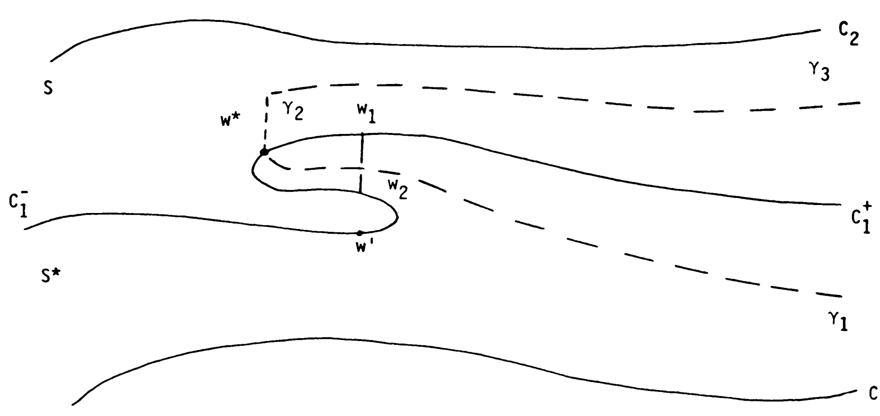

FIGURE 1

Now, let γ_1 be a piecewise analytic arc in S^* from w^* to $+\infty$ (see Figure 1). We may assume that $\sigma(u^*)$ intersects L at exactly one point and we let γ_2 be the segment of $\sigma(u^*)$ between w^* and that point. Then let γ_3 be the subarc of L from that point of intersection to $+\infty$. Then $\gamma_1 \cup \gamma_2 \cup \gamma_3$ is the boundary of a half strip domain A. Let C_1^+ be the open subarc of C_1 from w^* to $+\infty$ and let C_1^- be the open subarc from w^* to $-\infty$. Then $C_1^+ \subset A$ and $C_1^- \subset \mathbf{C} - \bar{A}$.

Next, pick $w'' \in C_1^+ \cap \Lambda_u$, and consider the segment $s = \overline{w''w'}$. Then s must cross $\gamma_1 \cup \gamma_2 \cup \gamma_3$ an odd number of times. It does not cross γ_2 at all and it cannot cross γ_3 an odd number of times, since w' and w'' are both below L. Thus some component l of $s \cap S^*$ must cross γ_1 an odd number of times, so that one endpoint of l, call it w_1, is on C_1^+. The other endpoint w_2 must be on ∂S^*, so it is not on C_2. Furthermore, the exterior wedge condition for Γ implies that S^* contains a parallel strip separating C_1 and C_2' so that s is above the strip and C_2' is below it. Thus $w_2 \notin C_2'$, and we conclude that $w_2 \in C_1^-$. But then Lemma 4 follows from Lemma 3 applied to Ω^*: Let ω_1 and ω_2 be the points on Γ corresponding to w_1 and w_2. Then 0, ω_1, ω^*, ω_2 satisfy the hypotheses of Lemma 3, so that (2.2) and (2.1) follow since $|\omega'| = |\omega_1|$.

3. Hölder continuity of the mapping functions.

PROOF OF THEOREM 1. We now assume that f maps D conformally onto Ω and $\bar{D}$ homeomorphically onto $\bar{\Omega}$. We must show that there exist positive δ and K such that, for any $\zeta_0 \in \partial D$ and $\zeta \in \partial D$ with $|\zeta - \zeta_0| < \delta$, we have

$$|f(\zeta) - f(\zeta_0)| < K|\zeta - \zeta_0|^{\alpha}.$$

We assume that $f(\zeta_0) = 0$ and that $f(-\zeta_0) = \omega_0$ where $|\omega_0| = (\operatorname{diam}\Gamma)/2 = R > 2r$ and r is so small that $|\omega_1 - \omega_2| < 2r$ implies that $\operatorname{diam} C(\omega_1, \omega_2) < R/2$. Now let $z = \operatorname{Log}((\zeta_0 + \zeta)/(\zeta_0 - \zeta))$ and $w = \log(1/\omega)$, for $\omega \in \Omega$. It then suffices to consider the mapping $w(z) = u(z) + iv(z)$ from $\Sigma = \{x + iy: |y| < \pi/2\}$ onto the half strip $\hat{S}$ which is the image of Ω under the logarithmic mapping. We then show that, for x_1 fixed and depending only on f and $x_2 > x_1$ with $z_2 = x_2 + iy_2 \in \partial\Sigma$, we have

$$\alpha x_2 - u(z_2) \leqslant M, \tag{3.1}$$

where M depends only on f. As in **[3]** we prove (3.1) by comparing modules of quadrilaterals.

On account of the α-wedge condition at $0 \in \Gamma$, the strip $\hat{S}$ contains a parallel half strip S' of width $\alpha\pi$. We may assume that $S' \supset \{w = u: u \geqslant u_0 = \log r\}$. For $u \geqslant u_0$, let $\sigma(u)$ be the maximal subarc of $\Lambda_u \cap \hat{S}$ which intersects S'. Now choose $x_1 = \max\{\operatorname{Re} z(w), w \in \sigma(u_0)\}$ so that x_1 depends only on f. For $x \geqslant x_1$, define

$$\gamma_x = \{w(z): \operatorname{Re} z = x, |y| \leqslant \pi/2\},$$
$$\bar{u}(x) = \sup\{u: \sigma(u) \cap \gamma_x \neq \varnothing\},$$

and

$$\underline{u}(x) = \inf\{u: \sigma(u) \cap \gamma_x \neq \varnothing\}.$$

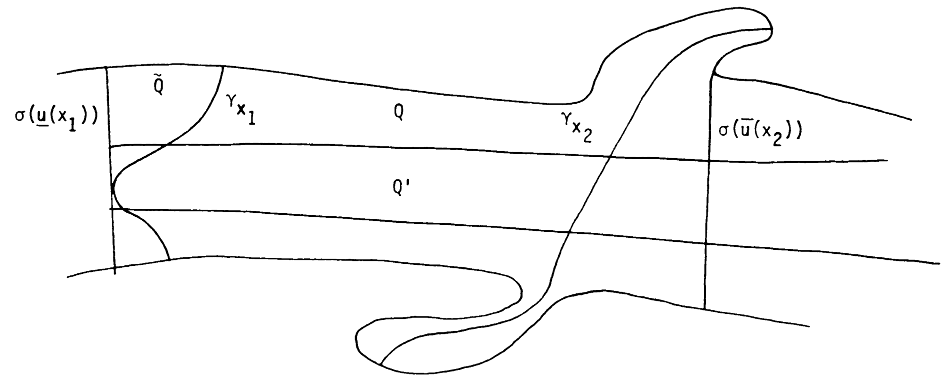

FIGURE 2

For $x_2 > x_1$, let $Q = \{w(z)\colon\ z = x + iy \in \Sigma,\ x_1 < x < x_2\}$, $Q' = \{w \in S'\colon \underline{u}(x_1) < u < \bar{u}(x_2)\}$ and $\tilde{Q}$ = the component of $\hat{S} - \sigma(\bar{u}(x_2)) - \sigma(\underline{u}(x_1))$ which contains Q' (see Figure 2). Q' is a rectangle, while Q and $\tilde{Q}$ are quadrilaterals, two of whose sides are "horizontal" subarcs of C_1 and C_2. Let $M(Q)$ be the module of the family of curves connecting the horizontal sides of Q with corresponding definitions for $M(Q')$ and $M(\tilde{Q})$. Then by conformal invariance and the comparison principle for modules,

$$\frac{1}{\pi}(x_2 - x_1) = M(Q) \leqslant M(\tilde{Q}) \leqslant M(Q') = \frac{\bar{u}(x_2) - \underline{u}(x_1)}{\alpha\pi}$$

so that

$$\alpha x_2 - \bar{u}(x_2) \leqslant \alpha x_1 - \underline{u}(x_1) \leqslant K_1$$

where K_1 depends on f. Now suppose that $z_2 = x_2 - i\pi/2$ and that $w(z_2) \in C_1$. Then

$$\alpha x_2 - u(z_2) \leqslant K_1 + \bar{u}(x_2) - \underline{u}(x_2) + \underline{u}(x_2) - u(z_2).$$

Since the width of S is at most 2π, one may use the Ahlfors distortion theorem (see [**3**, p. 317]) to show that

$$\bar{u}(x_2) - \underline{u}(x_2) \leqslant 4\pi.$$

It remains only to show that $\underline{u}(x_2) - u(z_2)$ is bounded. To see this let $w^* = u^* + iv^* \in \sigma(\underline{u}(x_2)) \cap C_1$. Let C_1^+ be the closed subarc of C_1 from w^* to $+\infty$. Let $u' = \min\{u\colon w = u + iv \in C_1^+\}$. Let $w' = u' + iv \in C_1^+$. Then γ_{x_2} is in the component of $\hat{S} - \sigma(\underline{u}(x_2))$ with $+\infty$ as a boundary point so that $\gamma_{x_2} \cap C_1 \in C_1^+$, and $u(z_2) \geqslant u'$. But then w' separates w^* from $+\infty$, and we may apply Lemma 2 to see that

$$\underline{u}(x_2) - u(z_2) \leqslant u^* - u' < \log M_2$$

and Theorem 1 is proved.

Proof of Theorem 2. We now suppose that Γ satisfies an exterior α-wedge condition and apply Theorem 1 to the exterior mapping $f^*\colon \{|\zeta| > 1\} = D^* \to \Omega^*$ to see that f^* is Hölder continuous on $\overline{D^*}$. We then apply Theorem 1 of [2] to see that f^{-1} is Hölder continuous with exponent $1/(2-\alpha)$ for approach in the "kernel". That is, considering without loss of generality $0 \in \Gamma$, there exists $K > 0$ such that

$$|f^{-1}(\omega) - f^{-1}(0)| < K|\omega|^{1/(2-\alpha)} \tag{3.2}$$

for ω on the crosscut of Ω at 0 as in the Remark to Lemma 4. We will establish (3.2) for any $\omega \in \Gamma$ with $|\omega| < \delta$, for some $\delta > 0$. We now let

$$z = \log((\zeta_0 + \zeta)/(\zeta_0 - \zeta)) \quad \text{and} \quad w = \log((\omega - \omega_0)/\omega)$$

and consider the mapping $z(w) = x(w) + iy(w)$ from S onto Σ where S is the infinite strip as in the proof of Lemma 4. Theorem 1 of [2] guarantees that for $w \in \sigma(u)$,

$$\frac{1}{(2-\alpha)} u - x(w) \leqslant M \tag{3.3}$$

for M depending only on f. We shall show that for any $w \in C_1$ with $u = \operatorname{Re} w > \log((R-r)/r) = u_0$, (3.3) holds with M replaced by M', which again depends only on f. So, let $u > u_0$ be given and $w = u + iv \in C_1$. Let $u^* = u - M_4'$ for M_4' as in (2.1). It follows from Lemma 4 that $w^* = \sigma(u^*) \cap C_1$ is separated from $+\infty$ by w. Then

$$\begin{aligned} \frac{1}{(2-\alpha)} u - x(w) &\leqslant \frac{1}{(2-\alpha)} u^* - x(w^*) + \frac{1}{(2-\alpha)}(u - u^*) + x(w^*) - x(w) \\ &\leqslant M + \frac{1}{(2-\alpha)} M_4' = M'. \end{aligned}$$

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