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ON C^*-EMBEDDING IN βN AND THE CONTINUUM HYPOTHESIS

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ABSTRACT. Let $\beta\mathbf{N}$ denote the Stone-Čech compactification of the natural numbers $\mathbf{N}$ with the discrete topology. It is shown that the continuum hypothesis holds iff for each pair X and Y of homeomorphic subspaces of $\beta\mathbf{N}$, X is C^*-embedded in $\beta\mathbf{N}$ iff Y is. Related questions concerning C^*-embedded subsets of $\beta\mathbf{N}$ are investigated assuming the hypothesis $2^{\aleph_0} < 2^{\aleph_1}$.

1. Introduction. All hypothesized topological spaces are assumed to be completely regular and Hausdorff. Thus "space" will mean "completely regular Hausdorff topological space". For undefined notation and terminology see [**GJ** or **Wa**].

Let S be a subspace of a space X. In general, the question of whether S is C^*-embedded in X depends not only on the topology of S but also on "how S is placed in X". In other words, a space X may contain homeomorphic subspaces S and T with S C^*-embedded in X and T not. For example, Q and $Q\setminus\{0\}$ are homeomorphic dense subspaces of βQ; the former is C^*-embedded in βQ, the latter is not (Q denotes the space of rationals and βQ its Stone-Čech compactification). Another example is provided by the homeomorphic subspaces $(-\infty, 0]$ and $(-1, 0]$ of $\mathbf{R}$.

The situation can be different when one considers subspaces of $\beta\mathbf{N}$. In fact, it is consistent with the usual axioms of set theory that whether a subspace X of $\beta\mathbf{N}$ is C^*-embedded in $\beta\mathbf{N}$ depends only on the topology of X. Specifically, the following theorem is 2.2 of [**Wo**]. Recall that a space X is *weakly Lindelöf* if, for each open cover $\mathscr{C}$ of X, there exists a countable subfamily $\mathscr{F}$ of $\mathscr{C}$ such that

$$X = \mathrm{cl}_X[\bigcup\{F: F \in \mathscr{F}\}].$$

We denote the continuum hypothesis by CH, and the cardinal $2^{\aleph_0}$ by c.

1.1 THEOREM. *Assume CH. Then the following conditions on a subspace X of $\beta\mathbf{N}$ are equivalent*:

(a) *X is C^*-embedded in $\beta\mathbf{N}$.*
(b) $|C^*(X)| = c$.
(c) *X is weakly Lindelöf.*

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However, if one assumes that $c = 2^{\aleph_1}$ (which also is consistent with the usual axioms of set theory), the situation is known to be different. Denote the discrete space of cardinality α by $D(\alpha)$ (thus $\mathbf{N} = D(\aleph_0)$). The following result is due to Efimov (see Remark 8 on p. 274 of **[E]**).

1.2 THEOREM. *Assume $c = 2^{\aleph_1}$. Then $\beta\mathbf{N}$ contains a C^*-embedded copy of $D(\aleph_1)$.*

By contrast, Balcar, Simon and Vojtaš prove the following result without using any set-theoretic assumptions (see 3.5 of **[BSV]**). (This result was independently proved (but not published) by K. Kunen and by S. Shelah.)

1.3 THEOREM. *$\beta\mathbf{N}$ contains a copy S of $D(\aleph_1)$ such that $\mathrm{cl}_{\beta\mathbf{N}} S$ is homeomorphic to the one-point compactification of the space $\{\alpha \in \beta D(\aleph_1)$: there exists $A \subset D(\aleph_1)$ such that $|A| \leqslant \aleph_0$ and $\alpha \in \mathrm{cl}_{\beta D(\aleph_1)} A\}$. In particular, S is not C^*-embedded in $\beta\mathbf{N}$.*

Thus if $c = 2^{\aleph_1}$, $\beta\mathbf{N}$ contains two homeomorphic subspaces, one C^*-embedded in $\beta\mathbf{N}$ and the other not.

1.4 DEFINITION. Let $\mathscr{P}$ be a topological property.

(a) A space X has the *absolute C^*-embedding property for $\mathscr{P}$* if, whenever S is a C^*-embedded subspace of X, S has $\mathscr{P}$, and T is a subspace of X that is homeomorphic to S, then T is C^*-embedded in X.

(b) A space X has the *absolute C^*-embedding property* if X has the absolute C^*-embedding property for $\mathscr{P}$ for every $\mathscr{P}$.

Thus $\beta\mathbf{N}$ has the absolute C^*-embedding property if CH is assumed, but does not have it if it is assumed that $c = 2^{\aleph_1}$. This raises the question of whether $\beta\mathbf{N}$ has the absolute C^*-embedding property if $c < 2^{\aleph_1}$. In §2 we show the answer is "no"; in fact, we prove the following, which is the main result of this paper.

1.5 THEOREM. *The following are equivalent:*

(a) *CH,*

(b) *$\beta\mathbf{N}$ has the absolute C^*-embedding property.*

In §3 we produce examples of some topological properties $\mathscr{P}$ such that $\beta\mathbf{N}$ has the absolute C^*-embedding property for $\mathscr{P}$ iff $c < 2^{\aleph_1}$.

2. C^*-embedding in $\beta\mathbf{N}$ when CH fails. In this section we prove 1.5. Recall that a space X is a *P-space* if its G_δ-sets are open. See **[GJ** or **Wa]** for basic information on these spaces; note particularly that P-spaces have an open base of clopen sets. Denote by $\mathscr{B}(X)$ the set of clopen subsets of a space X. The following theorem is implicitly stated and proved in §2 of **[DvM]**. We include a proof for completeness.

2.1 THEOREM. *Let X be a P-space for which $|\mathscr{B}(X)| \leqslant c$. Then βX can be embedded in $\beta\mathbf{N}$.*

PROOF. Note that if $|\mathscr{B}(X)| \leqslant c$ then $|\mathscr{B}(\beta X)| \leqslant c$. Since βX is zero-dimensional, standard "evaluation map" techniques show that βX can be embedded in $\{0,1\}^c$, where $\{0,1\}$ is the two-point discrete space. The argument in §2 of **[DvM]** then shows that βX can be embedded in the absolute $E(\{0,1\}^c)$ of $\{0,1\}^c$ (see **[Wa**, Chapter 10, or **Wo$_2$]** for a discussion of absolutes). Since $E(\{0,1\}^c)$ is separable and extremally disconnected (since $\{0,1\}^c$ is separable), it can be embedded in $\beta\mathbf{N}$ (see [**E**]). $\square$

2.2 DEFINITION. For each ordinal α, define $L(\alpha)$ to be the topological space whose underlying set is $\alpha + 1 \setminus \{\lambda \in \alpha + 1: \lambda$ is a limit ordinal of countable cofinality$\}$, and which has the subspace topology inherited from the order topology on $\alpha + 1$. (Here, as usual, $\alpha + 1$ is thought of as the set of ordinals no greater than α.)

The space $L(\omega_2)$ has been previously used—see [**vD** or **D**], for example—to solve problems similar to the ones discussed herein. We collect some known properties of $L(\omega_2)$ in the following

2.3 PROPOSITION. (a) $L(\alpha)$ *is a Lindelöf P-space for every* α (*the proof is identical to that indicated in* [**vD**] *for the case* $\alpha = \omega_2$).

(b) *Let* $T = L(\omega_2)\setminus\{\omega_2\}$. *Then* T *is a dense C-embedded subspace of* $L(\omega_2)$ *and* $\upsilon T = L(\omega_2)$ (9L *of* [**GJ**]).

(c) $|\mathscr{B}(T)| = |\mathscr{B}(L(\omega_2))| = c \cdot \aleph_2$ [**vD**].

We need a special case of the following, which is (as indicated below) an immediate consequence of known results.

2.4 LEMMA. *Let* α *and* β *be two ordinals. Then* $L(\alpha) \times L(\beta)$ *is a Lindelöf space.*

PROOF. If Y is a space, let Y_δ denote the space whose underlying set is that of Y, and for which the G_δ-sets of Y form an open base. It is easy to see that $L(\alpha) \times L(\beta)$ is homeomorphic to $[(\alpha + 1) \times (\beta + 1)]_\delta$, where $\alpha + 1$ and $\beta + 1$ are given the usual order topology. It is known that if Y is a compact scattered space, then Y_δ is Lindelöf; see, for example, p. 27 of [**M**]. Since $(\alpha + 1) \times (\beta + 1)$ is compact scattered, the lemma follows. $\square$

2.5 COROLLARY. $L(\omega_2) \times L(\omega_2)$ *is Lindelöf and* $|\mathscr{B}(L(\omega_2) \times L(\omega_2))| = c \cdot \aleph_2$.

PROOF. For the second claim, note that as $L(\omega_2) \times L(\omega_2)$ is Lindelöf, every clopen set of $L(\omega_2) \times L(\omega_2)$ is the union of countably many basic clopen sets of $L(\omega_2) \times L(\omega_2)$ of the form $A \times B$, where $A, B \in \mathscr{B}(L(\omega_2))$. Thus

$$\begin{aligned}|\mathscr{B}(L(\omega_2) \times L(\omega_2))| &\leqslant \left(|\mathscr{B}(L(\omega_2))| \times |\mathscr{B}(L(\omega_2))|\right)^{\aleph_0}\\ &= (c \cdot \aleph_2)^{\aleph_0} \quad \text{(follows from 2.3(c))}\\ &= c \cdot \aleph_2. \quad \square\end{aligned}$$

Henceforth we denote the space $L(\omega_2)$ by L.

PROOF OF 1.5. (a) $\Rightarrow$ (b). This is part of 1.1.

(b) $\Rightarrow$ (a). Suppose CH fails. Let $J = T \oplus T$ (the direct sum of two copies of the space T of 2.3(b)). Note that J is a P-space. By 2.3(c), $|\mathscr{B}(J)| = \aleph_2 \cdot c = c$ (as CH fails). Hence, by 2.1, J can be C^*-embedded in $\beta\mathbf{N}$.

Finite products of P-spaces are P-spaces (4K.6 of [**GJ**]), so, by 2.5, $L \times L$ is a P-space and $|\mathscr{B}(L \times L)| = c$. Hence, by 2.1, $\beta(L \times L)$ can be embedded in $\beta\mathbf{N}$. Now $(\{\omega_2\} \times L) \cup (L \times \{\omega_2\}) \setminus \{(\omega_2, \omega_2)\} = J_1$ is homeomorphic to J and is a subspace of $L \times L$. But (ω_2, ω_2) is in the $L \times L$ closure of the complementary clopen sets $\{\omega_2\} \times L$ and $L \times \{\omega_2\}$ of J_1, so J_1 is not C^*-embedded in $L \times L$. Thus a homeomorph of J can be embedded in $\beta\mathbf{N}$ in such a way that it is not C^*-embedded in $\beta\mathbf{N}$. Hence (b) fails. □

3. What happens when $c < 2^{\aleph_1}$. We now show that for certain topological properties $\mathscr{P}$, $\beta\mathbf{N}$ has the absolute C^*-embedding property for $\mathscr{P}$ iff $c < 2^{\aleph_1}$.

Recall (see [**B**]) that a space X is *$\delta\theta$-refinable* if, for each open cover $\mathscr{C}$ of X, there exists a countable collection $\{\gamma_n: n \in \mathbf{N}\}$ of open covers of X, each refining $\mathscr{C}$, such that for each $x \in X$ there exists $n(x) \in \mathbf{N}$ for which $|\{V \in \gamma_{n(x)}: x \in V\}| \leqslant \aleph_0$. A space is *$\aleph_1$-compact* if it has no uncountable closed discrete subsets. The following result of Aull appears in [$\mathbf{A}_1$].

3.1 THEOREM. *An $\aleph_1$-compact $\delta\theta$-refinable space is Lindelöf.*

An immediate consequence is the following

3.2 THEOREM. *The following are equivalent*:

(a) $c < 2^{\aleph_1}$.

(b) $\beta\mathbf{N}$ *has the absolute C^*-embedding property for "normal $\delta\theta$-refinable"*.

PROOF. (a) ⇒ (b). Let X and Y be homeomorphic normal $\delta\theta$-refinable subspaces of $\beta\mathbf{N}$. If X is $\aleph_1$-compact, then, by 3.1, X is Lindelöf. By 5.2 of [**N**], Lindelöf subspaces of F-spaces are C^*-embedded, so X and Y are C^*-embedded in $\beta\mathbf{N}$. If X is not $\aleph_1$-compact, X contains a closed copy of $D(\aleph_1)$. Since X is normal, this is C^*-embedded in X. Thus $|C^*(Y)| = |C^*(X)| \geqslant |C^*(D(\aleph_1))| = 2^{\aleph_1} > c = |C^*(\beta\mathbf{N})|$, so neither X nor Y is C^*-embedded in $\beta\mathbf{N}$.

(b) ⇒ (a). If $c = 2^{\aleph_1}$, then as noted in §1, $\beta\mathbf{N}$ contains two copies of $D(\aleph_1)$, one C^*-embedded and the other not. Since $D(\aleph_1)$ is normal $\delta\theta$-refinable, (b) fails. □

Note that the space T of 2.3(b) is normal, so "$\delta\theta$-refinable" cannot be dropped from 3.2(b) above.

Although not directly connected to our previous work, the following related results are of interest.

3.3 LEMMA. *Assume $c < 2^{\aleph_1}$. Then a normal, $\delta\theta$-refinable weakly Lindelöf space of weight no greater than c is Lindelöf.*

PROOF. It is a special case of 2.4 of [**CH**] that if X is weakly Lindelöf, then $|C^*(X)| \leqslant w(x)^{\aleph_0}$, where $w(x)$ denotes the weight of X (the least cardinality of an open base of X). Thus $|C^*(X)| = c$ as $w(x) \leqslant c$. Now argue as in the proof of 3.2. □

3.4 EXAMPLE. Assume that $c = 2^{\aleph_1}$. By 2.1, $\beta\mathbf{N} \setminus \mathbf{N}$ contains a C^*-embedded copy S of $D(\aleph_1)$. It follows that $\mathbf{N} \cup S$ is a normal (in fact, perfectly normal) separable θ-refinable space, but is not Lindelöf. Note that $w(\mathbf{N} \cup S) \leqslant c$ as $\mathbf{N} \cup S \subseteq \beta\mathbf{N}$. Thus the assumption "$c < 2^{\aleph_1}$" in 3.3 cannot be dropped.

Let $\mathscr{R}(X)$ denote the collection of regular closed subsets of X.

3.5 THEOREM. *The following are equivalent*:

(a) $c < 2^{\aleph_1}$.

(b) *If X is normal and $\delta\theta$-refinable and $|\mathscr{R}(X)| \leqslant c$, then X is Lindelöf.*

(c) *Each normal separable θ-refinable space is Lindelöf.*

PROOF. (a) ⇒ (b). If there were $\aleph_1$ pairwise disjoint nonempty open subsets of X, then $|\mathscr{R}(X)| \geqslant 2^{\aleph_1} > c$. Thus X satisfies the countable chain condition and hence is weakly Lindelöf (see 1.1 of [**Wo**], for example). Since X is regular, $w(X) \leqslant |\mathscr{R}(X)|$. Now apply 3.3.

(b) ⇒ (c). If X is separable, then $|\mathscr{R}(X)| \leqslant c$, since if T is dense in X, then $A \to \mathrm{cl}_X A$ is a bijection from $\mathscr{R}(T)$ onto $\mathscr{R}(X)$.

(c) ⇒ (a). Consider 3.4. □

The authors wish to thank the referee for suggesting Example 3.4 and the inclusion of 3.5(c).

Note that the space $D(\aleph_1)$ witnesses the fact that "weakly Lindelöf" cannot be deleted from the statement of 3.3. The space T of 2.3(b) witnesses that "$\delta\theta$-refinable" cannot be deleted from 3.5(b), while Isbell's space Ψ (see 5I of [**GJ**]), which is θ-refinable, witnesses that "normal" cannot be deleted from 3.5(b).

4. Open questions. It seems plausible that a stronger result that 3.2 is true. In particular, it is known (see [**Z**]) that a normal θ-refinable space must (in the absence of measurable cardinals, which obviously do not concern us here) be realcompact. This together with 3.2 suggest the following

4.1 *Question.* Assume that $c < 2^{\aleph_1}$. Does $\beta\mathbf{N}$ have the absolute C^*-embedding property for realcompactness? In fact, does there exist a realcompact C^*-embedded non-weakly-Lindelöf subspace of $\beta\mathbf{N}$?

Note that the example J used in the proof of 1.5 is not realcompact. We have the following partial result.

4.2 THEOREM. *Let X be a subspace of $\beta\mathbf{N}$ that is a realcompact P-space. The following are equivalent*:

(a) *Every subspace of $\beta\mathbf{N}$ that is homeomorphic to X is C^*-embedded in $\beta\mathbf{N}$.*

(b) *X is Lindelöf.*

PROOF. (b) ⇒ (a). This follows from 5.2 of [**N**] (as quoted in 3.2).

(a) ⇒ (b). Assume (b) fails. If $|C^*(X)| > c$, then (a) fails and we are done, so assume $|C^*(X)| = c$. Since X is a realcompact P-space but is not Lindelöf, it follows from Lemma 3D of [$\mathbf{A}_2$] that X contains complementary clopen subsets A and $X \setminus A$, neither of which is Lindelöf. Arguing as in the proof of 5.3 of [**DF**], we see that there exist compact subsets K and L of $\mathrm{cl}_{\beta X} A \setminus X$ and $\mathrm{cl}_{\beta X}(X \setminus A) \setminus X$, respectively, such that the quotient space Y of $X \cup K \cup L$, formed by collapsing the compact set $K \cup L$ to a point, is a P-space containing X as a dense subspace. Then $|C^*(Y)| = c$, so obviously $|\mathscr{B}(Y)| \leqslant c$. Hence, by 2.1, there exists a copy of βY in $\beta\mathbf{N}$. As $\mathrm{cl}_Y A \cap \mathrm{cl}_Y(X \setminus A) \neq \varnothing$, this copy of βY contains a copy of X that is not C^*-embedded in $\beta\mathbf{N}$. □

We are left with these problems:

4.3 *Questions.* (a) Suppose that $c < 2^{\aleph_1}$ and that X is a realcompact P-space that is C^*-embedded in $\beta\mathbf{N}$. Must X be Lindelöf? (Since a weakly Lindelöf P-space is easily seen to be Lindelöf, this is a special case of the second question in 4.1.)

(b) Suppose that X is a realcompact space (or even a realcompact P-space) and $|f[X]| \leqslant \aleph_0$ for every $f \in C^*(X)$. Must X be Lindelöf? (See **[LR]** for work related to this.)

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