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A HOLOMORPHIC FUNCTION WITH WILD BOUNDARY BEHAVIOR

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To the memory of Darja

ABSTRACT. Let B be the open unit ball in $\mathbf{C}^N$, $N > 1$. It is known that if f is a function holomorphic in B, then there are $x \in \partial B$ and an arc Λ in $B \cup \{x\}$, with x as one endpoint along which f is constant. We prove

THEOREM. *There exist an $r > 0$ and a function f holomorphic in B with the property that, if $x \in \partial B$ and Λ is a path with x as one endpoint, such that $\Lambda - \{x\}$ is contained in the open ball of radius r which is contained in B and tangent to ∂B at x, then $\lim_{z \in \Lambda, z \to x} f(z)$ does not exist.*

We denote by B the open unit ball in $\mathbf{C}^N$, $N > 1$. For each $x \in \partial B$ and r, $0 < r \leqslant 1$, let $D(x, r)$ be the open ball of radius r, contained in B and tangent to ∂B at x. We prove the following

THEOREM. *There exist an $r > 0$ and a function f holomorphic in B such that if $x \in \partial B$ and Λ is a path contained in $D(x, r)$, except for its endpoint x, then $\lim_{z \in \Lambda, z \to x} f(z)$ does not exist.*

It is known that r in the Theorem has to be strictly smaller than 1 [**2**]; whether or not it can be arbitrarily close to 1 is an open question.

For each $x \in \partial B$ and each ρ, $0 < \rho < 1$, let $H(\rho x)$ be the real hyperplane through ρx, tangent to ρB at ρx. If $R > 0$ let

$$W(\rho x, R) = \{ y \in H(\rho x) : |y - \rho x| < R \}.$$

Thus $W(\rho x, R)$ is the relatively open ball in $H(\rho x)$ of radius R centered at ρx.

LEMMA 1. *There is an $r > 0$ with the following property: let $0 < a < 1$; there exists $L \in \mathbf{N}$, numbers ρ_l, $1 \leqslant l \leqslant L$, $a < \rho_1 < \cdots < \rho_L < \rho_{L+1} = 1$, and numbers $R_l > 0$, $1 \leqslant l \leqslant L$, such that $\overline{W(x, R_l)} \subset \rho_{l+1} B$ for every $x \in \partial(\rho_l B)$, $1 \leqslant l \leqslant L$, and for each l, $1 \leqslant l \leqslant L$, there is a finite set $T_l \in \partial(\rho_l B)$ such that*

(i) *$\overline{W(x, R_l)} \cap H(y) = \varnothing$ whenever $x, y \in T_l$, $x \neq y$, $1 \leqslant l \leqslant L$;*

(ii) *given any $y \in \partial B$ there exist l, $1 \leqslant l \leqslant L$, and $z \in T_l$ such that if Λ is a path joining a point in aB with y, such that $\Lambda - \{y\} \subset D(y, r)$, then Λ meets $W(z, R_l)$.*

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LEMMA 2. *Let* $0 < a < 1$. *Let* L, ρ_l, R_l, *and* T_l, $1 \leqslant l \leqslant L$, *be as in Lemma* 1. *Given* $\varepsilon > 0$ *and* $C < \infty$ *there is a polynomial* P *such that*

(i) $\operatorname{Re} P > C$ *on* $\bigcup_{l=1}^{L} \bigcup_{x \in T_l} W(x, R_l)$;

(ii) $|P| < \varepsilon$ *on* aB.

PROOF. Choose ρ_l': $a < \rho_1' < \rho_1 < \rho_2' < \cdots < \rho_L' < \rho_L < \rho_{L+1}' < 1$ such that for each l, $1 \leqslant l \leqslant L$, $\overline{W(x, R_l)} \subset \rho_{l+1}' B$ $(x \in \partial(\rho_l B))$. Fix l, $1 \leqslant l \leqslant L$, and denote $W_l = \bigcup_{x \in T_l} \overline{W(x, R_l)}$. If $\delta_l > 0$ and $C_l < \infty$, then, by Lemma 1(i), one can prove, similarly to the proof of Theorem 4 in [**1**], that there is a polynomial P_l such that $|P_l| < \delta_l$ on $\rho_l' B$ and $\operatorname{Re} P_l > C_l$ on W_l. If we choose δ_l and C_l properly, then $P = \sum_{l=1}^{L} P_l$ will have all the required properties. This completes the proof.

PROOF OF THE THEOREM. By Lemmas 1 and 2 there exist an $r > 0$, a sequence a_n, $0 < a_1 < \cdots < 1$, $\lim a_n = 1$, and a sequence of sets W_n, $W_n \subset a_{n+1}B - a_n\overline{B}$, such that if $n \in \mathbf{N}$, $x \in \partial B$, and Λ is a path joining a point in $a_n B$ with x, which satisfies $\Lambda - \{x\} \subset D(x, r)$, then Λ meets W_n; moreover, for each $n \in \mathbf{N}$, $\delta_n > 0$, and $C_n < \infty$ there is a polynomial P_n such that $|P_n| < \delta_n$ on $a_n\overline{B}$ and $\operatorname{Re} P_n > C_n$ on W_n. If the sequence C_n is chosen inductively to increase to $+\infty$ fast enough, and if the sequence δ_n is chosen to decrease to 0 fast enough, then the series $\sum_{n=1}^{\infty}(-1)^n P_n$ converges uniformly on compacta in B to a function f holomorphic in B with the property: if $x \in \partial B$ and Λ is a path with x as one endpoint which satisfies $\Lambda - \{x\} \subset D(x, r)$, then

$$\limsup_{z \in \Lambda, z \to x} \operatorname{Re} f(z) = +\infty, \qquad \liminf_{z \in \Lambda, z \to x} \operatorname{Re} f(z) = -\infty.$$

This completes the proof.

To prove Lemma 1, we first prove three lemmas.

LEMMA 3. *Let* $x, y \in \partial B$ *and* $|x - y| > 2R/\rho$, *where* $0 < \rho < 1$ *and* $R > 0$. *Then* $\overline{W(\rho x, R)} \cap H(\rho y) = \varnothing$.

PROOF. Suppose $z \in \overline{W(\rho x, R)} \cap H(\rho y)$. Then $|z|^2 < \rho^2 + R^2$; i.e., $z \in \overline{W(\rho y, R)}$ and, consequently, $\rho|x - y| \leqslant |z - \rho x| + |z - \rho y| \leqslant 2R$, a contradiction.

LEMMA 4. *Let* $0 < r < 1$, $0 < \rho < 1$, *and* $0 < P < 2^{1/2}$. *Suppose* $x, y \in \partial B$, *and* $|x - y| < P(1 - \rho)^{1/2}$. *Then* x *and* y *both lie on the same side of* $H(\rho x)$. *Moreover*, $H(\rho x) \cap D(y, r) \subset W(\rho x, Q(1 - \rho)^{1/2})$, *where* $Q = (1 - r)P + (2r)^{1/2}$.

PROOF. The first statement follows from the fact that $P < 2^{1/2}$, which implies that $|x - y| < (2(1 - \rho))^{1/2}$. Suppose $H(\rho x) \cap D(y, r)$ is not empty. Write $y = \alpha x + w$, $\rho < \alpha \leqslant 1$, $|w|^2 + \alpha^2 = 1$. The center of $D(y, r)$ is at a distance of $|\rho - (1 - r)\alpha|$ from $H(\rho x)$ and at a distance of $(1 - r)(1 - \alpha^2)^{1/2}$ from $\mathbf{R}x$. Consequently, $H(\rho x) \cap D(y, r) \subset W(\rho x, R)$, where

$$\begin{aligned} R &= (1 - r)(1 - \alpha^2)^{1/2} + \left\{r^2 - [\rho - (1 - r)\alpha]^2\right\}^{1/2} \\ &= (1 - r)(1 - \alpha^2)^{1/2} + \{[(1 - \rho) - (1 - r)(1 - \alpha)] \\ &\qquad \cdot [2r - (1 - \rho) + (1 - r)(1 - \alpha)]\}^{1/2}. \end{aligned}$$

Since $|x - y| < P(1-\rho)^{1/2}$, we have $(1-\alpha)^2 + (1-\alpha^2) < P^2(1-\rho)$; hence $1 - \alpha < (1-\rho)P^2/2$, and, consequently,

$$R \leqslant (1-r)(1-\rho)^{1/2}(P/2^{1/2}) \cdot 2^{1/2} + (2r(1-\rho))^{1/2}.$$

This completes the proof.

LEMMA 5. *Let $p \in \mathbf{N}$ and $x \in \partial B$. There exist a neighbourhood $U \subset \partial B$ of x, an $r_0 > 0$, and $M \in \mathbf{N}$ such that, for any r, $0 < r < r_0$, there are finite sets $S_m \subset U$, $1 \leqslant m \leqslant M$, such that $U \subset \bigcup_{m=1}^{M} \bigcup_{y \in S_m}(y + rB)$ and $|y - z| \geqslant pr$ whenever y, $z \in S_m$, $y \neq z$, $1 \leqslant m \leqslant M$.*

PROOF. *Part* 1. We prove the following. Let $W \subset \mathbf{R}^{2N-1}$ be a bounded set and let $k \in \mathbf{N}$. There is a $\mu = \mu(k, N) \in \mathbf{N}$ such that, given any $r > 0$, there are finite sets $T_m \subset \mathbf{R}^{2N-1}$, $1 \leqslant m \leqslant \mu$, such that $W \subset \bigcup_{m=1}^{\mu} \bigcup_{y \in T_m}(y + rB)$ (in this part B is the open unit ball in $\mathbf{R}^{2N-1}$) and $|y - z| \geqslant kr$ whenever y, $z \in T_m$, $y \neq z$, $1 \leqslant m \leqslant \mu$.

To do this put $L = 2N - 1$, choose $q \in \mathbf{N}$ such that $q > kL^{1/2}$, and put $\mu = q^L$. Let $r > 0$. Define $S \subset \mathbf{R}^L$ by

$$S = \{kr(s_1, s_2, \ldots, s_L) : s_i \in \mathbf{Z}, 1 \leqslant i \leqslant L\}.$$

Observe that $|y - z| \geqslant kr$ whenever y, $z \in S$, $y \neq z$. Further, let P be the set of μ points in the cube $\{t \in \mathbf{R}^L : 0 \leqslant t_i \leqslant kr, 1 \leqslant i \leqslant L\}$, defined by

$$P = \{(kr/q)(s_1, s_2, \ldots, s_L) : s_i \in \mathbf{Z}, 1 \leqslant s_i \leqslant q, 1 \leqslant i \leqslant L\}.$$

There are μ sets of the form $y + S$, where $y \in P$, and they have the following properties:

(a) if $y \in P$ and $z, w \in y + S$, $z \neq w$, then $|w - z| \geqslant kr$;

(b) $\mathbf{R}^L = \bigcup_{y \in P} \bigcup_{z \in y+S}(z + K)$,

where K is the cube $\{t \in \mathbf{R}^L : |t_i| < kr/q, 1 \leqslant i \leqslant L\}$.

Since $q > kL^{1/2}$, it follows that $kr/q < rL^{-1/2}$; hence $K \subset rB$, which implies $\mathbf{R}^L = \bigcup_{y \in P} \bigcup_{z \in y+S}(z + rB)$. Now the assertion follows from the boundedness of W.

Part 2. There exist an open neighbourhood $U' \subset \partial B$ of x, an open neighbourhood $V \subset \mathbf{R}^{2N-1}$ of 0, a constant $c > 0$, and a map Ψ from V onto U' such that

$$(1/c)|u - v| < |\Psi(u) - \Psi(v)| < c|u - v| \qquad (u, v \in V).$$

Let $U \subset U'$ be a compact neighbourhood of x. The statement of the lemma now follows easily from Part 1. This completes the proof.

PROOF OF LEMMA 1. It is enough to prove the following. Let $x \in \partial B$. There exist $M \in \mathbf{N}$, $r > 0$, a neighborhood $U \subset \partial B$ of x, and a, $0 < a < 1$, such that the following holds: Given any ρ_1, $a < \rho_1 < 1$, there exist $R > 0$ and ρ_m, $1 < m \leqslant M + 1$, $\rho_1 < \rho_2 < \cdots < \rho_M < \rho_{M+1} < 1$, such that $\overline{W(y, r)} \subset \rho_{m+1}B$ $(y \in \partial(\rho_m B)$, $1 \leqslant m \leqslant M)$ and, for each m, $1 \leqslant m \leqslant M$, there is a finite set $S_M \subset \partial(\rho_m B)$ such that

(i) $\overline{W(y, R)} \cap H(z) = \varnothing$ whenever y, $z \in S_m$, $y \neq z$, $1 \leqslant m \leqslant M$;

(ii) given any $y \in U$ there exist m, $1 \leqslant m \leqslant M$, and $z \in S_m$ such that if Λ is a path joining a point in $\rho_1 B$ with y, where $\Lambda - \{y\} \subset D(y, r)$, then Λ meets $W(z, R)$.

To prove this let $p = 9$ and let U, r_0, and M be as in Lemma 5. Choose P, $0 < P < 2^{1/2}$, and $r > 0$ such that

$$Q = (1 - r)P + (2r)^{1/2} < (2M)^{-1/2} \tag{1}$$

and

$$8(1 - r) + (2r)^{1/2}/P < p. \tag{2}$$

Note that, by (2),

$$8Q/P < p. \tag{3}$$

Choose $a < 1$ so close to 1 that

$$1/2 < a, \tag{4}$$

$$1 - a < r, \tag{5}$$

and

$$P((1 - a)/2)^{1/2} < r_0. \tag{6}$$

Let $a < \rho_1 < 1$. Set $\vartheta = (1 - \rho_1)/(2M)$ and let

$$\rho_m = \rho_1 + (m - 1)\vartheta \qquad (1 \leqslant m \leqslant M + 1).$$

Put $R = Q(1 - \rho_1)^{1/2}$. By (1) and (4),

$$\begin{aligned} R &< (2M)^{-1/2}(1 - \rho_1)^{1/2} = \vartheta^{1/2} = (\rho_{m+1} - \rho_m)^{1/2} \\ &\leqslant \left(\rho_{m+1}^2 - \rho_m^2\right)^{1/2} \qquad (1 \leqslant m \leqslant M), \end{aligned}$$

so $\overline{W(y, R)} \subset \rho_{m+1}B$ $(x \in \partial(\rho_m B), 1 \leqslant m \leqslant M)$.

Now let $\varepsilon = P(1 - \rho_1)^{1/2} \cdot 2^{-1/2}$. By (6), $\varepsilon < r_0$. Furthermore, since $(1 - \rho_1)/2 < 1 - \rho_m$ $(1 \leqslant m \leqslant M)$, it follows that $\varepsilon < P(1 - \rho_m)^{1/2}$ $(1 \leqslant m \leqslant M)$. By (5), $1 - \rho_m < r$ $(1 \leqslant m \leqslant M)$; since $P < 2^{1/2}$, it follows, by Lemma 4, that if y, $z \in \partial B$, $|y - z| < \varepsilon$, then both y and z lie on the same side of $H(\rho_m y)$, $1 \leqslant m \leqslant M$, and furthermore,

(7) if $1 \leqslant m \leqslant M$ and if $y, z \in \partial B$, $|y - z| < \varepsilon$, then every path Λ, which joins a point in $\rho_m B$ with z and satisfies $\Lambda - \{z\} \subset D(z, r)$, meets $W(\rho_m y, Q(1 - \rho_m)^{1/2}) \subset W(\rho_m y, R)$.

Furthermore, since $\varepsilon < r_0$, it follows by Lemma 5 that there are finite sets $T_m \subset \partial B$, $1 \leqslant m \leqslant M$, such that

(a) $|y - z| \geqslant p\varepsilon$ whenever $y, z \in T_m$, $y \neq z$, $1 \leqslant m \leqslant M$;

(b) $U \subset \bigcup_{m=1}^{M} \bigcup_{y \in T_m} (y + \varepsilon B)$.

Define $S_m = \rho_m T_m$ $(1 \leqslant m \leqslant M)$.

Suppose $1 \leqslant m \leqslant M$ and $y, z \in T_m$, $y \neq z$. By (a) and (3) it follows that

$$\begin{aligned} |y - z| &\geqslant p\varepsilon > 8(Q/P)\varepsilon = 8(Q/P) \cdot P((1 - \rho_1)/2)^{1/2} \\ &= 8Q \cdot 2^{-1/2}(1 - \rho_1)^{1/2} > 4R. \end{aligned}$$

By (4), $\rho_m > 1/2$ so, by Lemma 3, $\overline{W(\rho_m y, R)} \cap H(\rho_m z) = \varnothing$. This proves (i). Suppose $y \in U$. By (b) there exist m, $1 \leqslant m \leqslant M$, and $z \in T_m$ such that $|y - z| < \varepsilon$. Consequently, (ii) follows from (7). This completes the proof.

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