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NONUNIQUENESS OF THE CARDINALITY ATTACHED TO HOMOGENEOUS AW*-ALGEBRAS

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Dedicated to Professor H. Umegaki on his 60th birthday

ABSTRACT. It is proved that for any pair of infinite cardinal numbers α and β, there exists a homogeneous AW*-algebra that is both α-homogenous and β-homogeneous. This negatively settles a long-standing unsolved problem of type I AW*-algebras.

One of the long-standing open problems in the structure theory of AW*-algebras is the uniqueness of the cardinal number attached to a homogeneous AW*-algebra [**1**, pp. 88, 111; and Excercise 10, p. 118]. The problem was posed by Kaplansky [**5**, p. 460] and conjectured negatively in [**6**, p. 843]. In this note, we shall settle this problem negatively by proving the following theorem. Our method of proof is inspired by P. J. Cohen's forcing method developed in the field of mathematical logic (cf. [**3**]).

THEOREM. *For any pair of infinite cardinal numbers α and β, there exists a homogeneous* AW*-*algebra that is both α-homogeneous and β-homogeneous.*

By [**6**, Theorem 7], the algebra of bounded Z-linear operators on an $\aleph$-homogeneous AW*-module over a commutative AW*-algebra Z is an $\aleph$-homogeneous AW*-algebra. Thus we need only construct an AW*-module that is both α-homogeneous and β-homogeneous.

We can suppose that $\alpha < \beta$. Let A be a set with cardinality α, and B a set with cardinality β. Let P be the set of all one-to-one functions p such that

$$\operatorname{dom}(p) \subset A, \quad \operatorname{ran}(p) \subset B \quad \text{and} \quad \operatorname{card}(\operatorname{dom}(p)) < \alpha.$$

If $p \in P$, then $\operatorname{dom}(p)$ is a proper subset of A and, since $\alpha < \beta$, $\operatorname{ran}(p)$ is a proper subset of B.

For any $p \in P$, let $[p]$ be the set of all $q \in P$ such that q is an extension of p. Thus $[p] \cap [p] \neq \varnothing$ if and only if p and q have a common extension that is one-to-one.

Let $\mathscr{P} = \{[p]: p \in P\}$. It is easily seen that for $p, q \in P$, $[p] \cap [q]$ is either empty or of the form $[r]$ (r the minimal common extension of p, q). Let $\mathscr{U}$ be the topology on P generated by $\mathscr{P}$; by the preceding remark, $\mathscr{U}$ consists of the set of all unions $\bigcup [p_i]$, where (p_i) is any family in P.

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Let $\mathscr{B}$ be the set of all regular open sets in P, that is, the set of all open sets U in P such that $U = \operatorname{int}(\overline{U})$ [**2**, p. 13]. With suitable lattice operations, $\mathscr{B}$ is a complete Boolean algebra [**2**, Lemma 1, p. 25].

LEMMA 1. $\mathscr{P} \subset \mathscr{B}$.

PROOF. Let $p \in P$ and write $U = [p]$; then U is an open set by the definition of the topology of P, and $U \subset \operatorname{int}(\overline{U})$. Suppose $q \in \operatorname{int}(\overline{U})$, say $q \in [r] \subset \overline{U}$; we are to show that $q \in U$. If $s \in [r]$ then every neighborhood of s intersects $U = [p]$; thus if $t \in P$ is such that $s \in [t]$ then $[t] \cap [p] \neq \varnothing$; in other words, if $s, t \in P$ are such that s is a common extension of r, t, then t, p have a common extension in P. Summarizing (all elements are in P):

(i) q extends r;

(ii) if r, t have a common extension in P, then so do t, p.

From this, we are to infer that q extends p.

First, $\operatorname{dom}(p) \subset \operatorname{dom}(q)$. For, let $a \in \operatorname{dom}(p)$ and assume to the contrary that $a \notin \operatorname{dom}(q)$. Choose $b \in B$ with $b \notin \operatorname{ran}(p) \cup \operatorname{ran}(q)$ (possible since $\beta > \alpha$). Let t be the extension of q obtained by defining $t(a) = b$. Then t extends r by (i); therefore, t, p have a common extension by (ii); in particular, $t(a) = p(a)$, so $b = p(a) \in \operatorname{ran}(p)$, a contradiction.

Now let t be the restriction of q to $\operatorname{dom}(p)$. Then r, t have a common extension in P (namely q), and therefore so do t, p; it follows that $t = p$; thus q extends p, that is, $q \in U$. □

Let T be the Stone representation space of $\mathscr{B}$, $Z = C(T)$ the algebra of continuous complex-valued functions on T; thus T is an extremally disconnected compact space [**2**, Theorem 10, p. 92] and Z is a commutative AW*-algebra [**1**, p. 40]. We can identify $\mathscr{B}$ with the complete Boolean algebra of projections of Z; under this identification, the intersection $[p] \cap [q]$ becomes the product $[p][q]$ of the corresponding projections.

For each pair $a \in A$, $b \in B$, define

$$U(a, b) = \sup\{[p] : p(a) = b\},$$

the supremum being taken in the projection lattice of Z; thus, in the notation of [**2**, p. 25], $U(a, b) = (\bigcup_{p(a)=b}[p])^{\perp\perp}$ as a subset of P.

LEMMA 2. *If $a, a' \in A$ with $a \neq a'$, then $U(a, b)U(a', b) = 0$ for all $b \in B$.*

PROOF. Let $b \in B$ and suppose $p \in P$, $q \in P$ with $p(a) = b$, $q(a') = b$. Since $p(a) = b = q(a')$, no common extension of p, q can be one-to-one; therefore $[p] \cap [q] = \varnothing$. Thus $[p][q] = 0$ in Z. Since $U(a, b)$ is the supremum of such projections $[p]$, it follows that $U(a, b)[q] = 0$; similarly, varying q, $U(a, b)U(a', b) = 0$. □

LEMMA 3. *For each $b \in B$, $\sup_{a \in A} U(a, b) = 1$.*

PROOF. Let $b \in B$. Assuming to the contrary that $1 - \sup_{a \in A} U(a, b) \neq 0$, choose $p \in P$ with $[p] \leqslant 1 - \sup_{a \in A} U(a, b)$ (recall that $\mathscr{P}$ is a base for the topology of P). Then $[p] \cap U(a, b) = \varnothing$ for all $a \in A$. If $b \in \operatorname{ran}(p)$, say $b = p(a)$, then $[p] \subset U(a, b)$, contrary to $[p] \cap U(a, b) = \varnothing$. If $b \notin \operatorname{ran}(p)$, choose $a \in A \setminus \operatorname{dom}(p)$ and let q be the extension of p obtained by defining $q(a) = b$; then $q \in [p] \cap U(a, b)$, again a contradiction. □

LEMMA 4. *If $b, b' \in B$ with $b \neq b'$, then $U(a, b)U(a, b') = 0$ for all $a \in A$.*

PROOF. Let $a \in A$ and suppose $p \in P$, $q \in P$ with $p(a) = b$, $q(a) = b'$. Then $p(a) \neq q(a)$, so p and q can have no common extension; therefore $[p] \cap [q] = \varnothing$, that is, $[p][q] = 0$. Varying p, $U(a, b)[q] = 0$; varying q, $U(a, b)U(a, b') = 0$. □

LEMMA 5. *For each $a \in A$, $\sup_{b \in B} U(a, b) = 1$.*

PROOF. Let $a \in A$ and assume to the contrary that the indicated supremum is not 1. As in Lemma 3, choose $p \in P$ with $[p]U(a, b) = 0$ for all $b \in B$. If $a \in \operatorname{dom}(p)$, let $b = p(a)$; then $[p] \subset U(a, b)$, contrary to $[p] \cap U(a, b) = \varnothing$. If $a \notin \operatorname{dom}(p)$, choose $b \in B \setminus \operatorname{ran}(p)$ and let q be the extension of p obtained by defining $q(a) = b$; then $q \in [p] \cap U(a, b)$, again a contradiction. □

PROOF OF THE THEOREM. Let H be a β-homogeneous AW*-module over Z [**6**, Theorem 2] and let $(f_b)_{b \in B}$ be an orthonormal basis of H with cardinality β. For each $a \in A$, define $e_a \in H$ by the formula

$$e_a = \sum_{b \in B} U(a, b) f_b,$$

where the indicated sum exists by Lemmas 4 and 5 [**6**, Postulate (b), p. 842]. We shall show that the family $(e_a)_{a \in A}$ is also an orthonormal basis of H (with cardinality α). For each $a \in A$, we have [**6**, Lemma 2]

$$\begin{aligned}(e_a, e_a) &= \sum_{b, b' \in B} U(a, b)U(a, b')(f_b, f_{b'}) \\ &= \sum_{b \in B} U(a, b) = 1\end{aligned}$$

by Lemmas 4 and 5; and, for $a, a' \in A$ with $a \neq a'$, we have

$$\begin{aligned}(e_a, e_{a'}) &= \sum_{b, b' \in B} U(a, b)U(a', b')(f_b, f_{b'}) \\ &= \sum_{b \in B} U(a, b)U(a', b) = 0\end{aligned}$$

by Lemma 2. Thus $(e_a)_{a \in A}$ is an orthonormal family.

It remains only to show that the family is total in the appropriate sense [**6**, p. 843]. Suppose $x \in H$ is such that $(x, e_a) = 0$ for all $a \in A$. Then

$$\begin{aligned}0 &= \sum_{a \in A} (x, e_a)(e_a, x) \\ &= \sum_{a \in A;\, b, b' \in B} U(a, b)U(a, b')(x, f_b)(f_{b'}, x) \\ &= \sum_{a \in A, b \in B} U(a, b)(x, f_b)(f_b, x) \\ &= \sum_{b \in B} \Big(\sum_{a \in A} U(a, b) \Big)(x, f_b)(f_b, x) \\ &= \sum_{b \in B} (x, f_b)(f_b, x) \quad \text{(by Lemmas 2 and 3)} \\ &= (x, x) \quad \text{(by [6, Lemma 10])};\end{aligned}$$

therefore $x = 0$.

Thus H has orthonormal bases $(e_a)_{a \in A}$ and $(f_b)_{b \in B}$ of cardinalities α and β, respectively.

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