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PULL-BACKS FOR PROJECTORS IN FINITE GROUPS

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ABSTRACT. We exhibit a pull-back construction for Schunck class projectors in finite groups and some of its consequences.

In [3] a pull-back construction for projectors of saturated formations in finite soluble groups has been given[1], based on a result of Huppert that does not generalise to Schunck classes. Therefore (and with a specific application in mind—see [2]), in the present note we give an alternative approach to this construction and show that, in fact, pull-backs exist for Schunck class projectors in arbitrary finite groups. Here we rely on the theory of Schunck classes and their projectors in finite groups as developed in [1] and assume the reader to be familiar with terminology as well as main results of [1] mentioning only that, in [1, 4.2], we have shown that, for every Schunck class $\mathscr{H}$ and any finite group G, the set

$$\mathrm{Proj}^+_{\mathscr{H}}(G) = \left\{ \begin{array}{l} H \leqslant G \mid H \text{ is an } \mathscr{H}\text{-projector of } HN_m \\ \text{whenever } N_1 \trianglelefteq G,\ N_{i+1} \trianglelefteq HN_i\ (i = 1, \ldots, m-1) \end{array} \right\}$$

is nonempty; observe that $H \in \mathrm{Proj}^+_{\mathscr{H}}(G)$ and $N \trianglelefteq G$ implies that $H \in \mathrm{Proj}^+_{\mathscr{H}}(HN)$.

THEOREM. *Let H be a subdirect subgroup of $G_1 \times G_2$ and put $K = (H \cap G_1) \times (H \cap G_2)$. Further, let $X_i \in \mathrm{Proj}^+_{\mathscr{H}}(G_i)$ $(i = 1, 2)$ and let H_i be the inverse image of X_i in H with respect to the projection map π_i: $H \to G_i$. Assume that $H_1K = H_2K$ and put $X = H_1 \cap H_2$. Then for each $Y \in \mathrm{Proj}^+_{\mathscr{H}}(X)$ we have*

(i) *$Y \in \mathrm{Proj}^+_{\mathscr{H}}(H)$, and*

(ii) *X and Y are subdirect subgroups of $X_1 \times X_2$.*

PROOF. By definition of H_i, $H \cap G_{3-i} \leqslant H_i$ $(i = 1, 2)$. Therefore, applying Dedekind's modular law and using $H_1K = H_2K$, we get

$$\begin{aligned} X(H \cap G_{3-i}) &= (H_1 \cap H_2)(H \cap G_{3-i}) = H_i \cap [H_{3-i}(H \cap G_{3-i})] \\ &= H_i \cap [H_{3-i}(H \cap G_i)(H \cap G_{3-i})] = H_i \cap H_{3-i}K = H_i \cap H_iK = H_i. \end{aligned}$$

Consequently,

$$\begin{aligned} X/X \cap G_{3-i} &= X/X \cap (H \cap G_{3-i}) \cong X(H \cap G_{3-i})/(H \cap G_{3-i}) \\ &= H_i/(H \cap G_{3-i}) \cong X_i \in \mathscr{H}, \end{aligned}$$

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[1] Perhaps, we may remark that this construction is superfluous for the applications given in [3], in constrast to what is done below.

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as $H \cap G_{3-i}$ is the kernel of $\pi_i|_{H_i}$. Hence $Y \in \mathrm{Proj}^+_{\mathscr{H}}(X)$ covers both $X/X \cap G_1$ and $X/X \cap G_2$. Moreover,

$$XG_i = X(H \cap G_i)G_i = H_{3-i}G_i = X_{3-i} \times G_i,$$

and so X is a subdirect subgroup of $X_1 \times X_2$. In view of $X = Y(X \cap G_i)$ it is clear that Y is also subdirect in $X_1 \times X_2$, which completes the proof of (ii).

In order to prove (i), we proceed by induction on $|G|$. Applying [**1**, 3.3], we see that, w.l.o.g., $H = H_1K = H_2K$; note that $H_1K = H_2K \leqslant (X_1 \times X_2)K = X_1(H \cap G_1) \times X_2(H \cap G_2)$, which is a consequence of $H_i = X(H \cap G_{3-i})$ and $K = (H \cap G_1) \times (H \cap G_2)$, and that $H_iK/K \in \mathrm{Proj}^+_{\mathscr{H}}(H/K)$, which follows from $H/K \cong (G_1 \times G_2)/(H \cap G_i) \times G_{3-i}$ via the canonical map, as $X_i \in \mathrm{Proj}^+_{\mathscr{H}}(G_i)$. In addition, we may assume that $G_i = X_i \in \mathscr{H}$ $(i = 1, 2)$:

Indeed, suppose that $G_1 \notin \mathscr{H}$. Then [**1**, 2.1b] yields existence of $N \trianglelefteq G_1$ such that $G_1/N \in b(\mathscr{H})$. Aiming at an application of the inductive hypothesis, we want to replace G_1, G_2, H, K, X_1, X_2, H_1, H_2, X, Y by X_1N, G_2, $H \cap G^*$, $K \cap G^*$, X_1, X_2, H_1, $H_2 \cap G^*$, X, Y, where $G^* = (X_1N) \times G_2$. We check that the hypotheses carry over to these groups:

From $G_1 \times G_2 = G_1H = X_1(H \cap G_1)H = X_1H$—here $G_1/H \cap G_1 \cong G_1 \times G_2/(H \cap G_1) \times G_2 = HG_2/KG_2 \cong H/H \cap KG_2 = H/K = H_iK/K \in \mathscr{H}$ and $X_1 \in \mathrm{Proj}^+_{\mathscr{H}}(G_1)$ have been used—we obtain that

$$(X_1N)(H \cap X_1NG_2) = N(X_1H \cap X_1NG_2) = X_1N \times G_2,$$

while $X_1N \times G_2 = (H \cap X_1NG_2)G_2$ is immediate from $HG_2 = G_1 \times G_2$. Hence $H \cap X_1NG_2$ is subdirect in $X_1N \times G_2$. Recall that $H_i = X(H \cap G_{3-i}) \leqslant X_i \times G_{3-i}$. Therefore, using the defining equation for K and the fact that $X \leqslant (X_1 \times G_2) \cap (G_1 \times X_2) = X_1 \times X_2$, we calculate

$$\begin{aligned}(H_2 \cap X_1NG_2)(K \cap X_1NG_2) &= (X(H \cap G_1) \cap X_1NG_2)(H \cap X_1N)(H \cap G_2)\\ &= X(H \cap G_1 \cap X_1NG_2)(H \cap X_1N)(H \cap G_2)\\ &= X(H \cap X_1N)(H \cap G_2) = H_1(K \cap X_1NG_2);\end{aligned}$$

note that $H_1 \leqslant X_1 \times G_2$. By [**1**, 3.3], $X_1 \in \mathrm{Proj}^+_{\mathscr{H}}(X_1N)$.

Now by induction,

$$(*) \qquad Y \in \mathrm{Proj}^+_{\mathscr{H}}(H \cap X_1NG_2).$$

Let M/N be a minimal normal subgroup of G_1/N. Since $G_1/N \in b(\mathscr{H})$, $G_1/M \in \mathscr{H}$; i.e., $G_1 = X_1M$. As was observed earlier in this proof, $X = Y(X \cap G_2)$ and $XG_2 = X_1 \times G_2$, and thus $X_1G_2 = XG_2 = YG_2$. We conclude that $G_1G_2 = X_1MG_2 = YMG_2$, and

$$H = YMG_2 \cap H = Y(H \cap MG_2) = (H \cap X_1NG_2)(H \cap MG_2),$$
$$H \cap X_1NG_2 = Y(H \cap MG_2 \cap X_1NG_2).$$

In view of the canonical isomorphism from $H/H \cap G_2$ to $G_1G_2/G_2, \cong G_1$, we may apply [**1**, 4.1] to obtain from this decomposition of H

$$(**) \qquad H \cap X_1NG_2/H \cap NG_2 \in \mathrm{Proj}^+_{\mathscr{H}}(H/H \cap NG_2).$$

Combining (*) and (**) with [**1**, 3.3] shows that $Y \in \mathrm{Proj}^+_{\mathscr{H}}(H)$. This proves our claim that w.l.o.g., $G_i \in \mathscr{H}$ $(i = 1, 2)$. In particular, $G_i = X_i$, and so the definitions of H_i and X yield that $H_1 = H_2 = H = H_1 \cap H_2 = X$, in which case (i) holds trivially. □

We do not know whether the above result is true with $\mathrm{Proj}_{\mathscr{H}}$ instead of $\mathrm{Proj}^+_{\mathscr{H}}$—our proof depends crucially on working with elements of $\mathrm{Proj}^+_{\mathscr{H}}$ rather than arbitrary $\mathscr{H}$-projectors.

Now define

$$X(\mathscr{H}, \mathscr{F}) = \{G \in \mid \mathrm{Proj}^+_{\mathscr{H}}(G) \subseteq \mathscr{F}\},$$

where $\mathscr{H}$ is a Schunck class and $\mathscr{F}$ is a formation. Then $X(\mathscr{H}, \mathscr{F})$ is obviously a formation, provided only that $\mathscr{F}$ is.

When dealing with insoluble groups, however, it is sometimes more appropriate to employ the class $X^*(\mathscr{H}, \mathscr{F})$ of all finite groups H satisfying

(i) $\mathrm{Proj}^+_{\mathscr{H}}(H) \cap \mathscr{F} \neq \varnothing$, and

(ii) $N \trianglelefteq H$, $E \leqslant H$, $E/N \in \mathrm{Proj}^+_{\mathscr{H}}(H/N) \cap \mathscr{F} \Rightarrow E = FN$ for some $F \in \mathrm{Proj}^+_{\mathscr{H}}(H) \cap \mathscr{F}$:

this is due to the fact that H may have several conjugacy classes of $\mathscr{H}$-projectors, among which there can occur nonisomorphic ones. Of course, one is mainly interested in condition (i), but (ii) ensures that $X^*(\mathscr{H}, \mathscr{F})$ is a formation. The following consequence of [**1**, 3.3] indicates that condition (ii) makes sense.

LEMMA. *If X denotes one of* $\mathrm{Proj}_{\mathscr{H}}$, $\mathrm{Proj}^+_{\mathscr{H}}$, *then* $X(G/N) = X(G)N/N$ *whenever* $N \trianglelefteq G$.

COROLLARY. $X^*(\mathscr{H}, \mathscr{F})$ *is a formation.*

PROOF. Quotient closure of $\mathscr{X} = X^*(\mathscr{H}, \mathscr{F})$ is trivial. Therefore, to verify subdirect product closure, it suffices to consider groups H with two distinct minimal normal subgroups N_1 and N_2 such that $H/N_i \in \mathscr{X}$ $(i = 1, 2)$; it is to be shown that $H \in \mathscr{X}$.

Put $G_i = H/N_{3-i}$, so that (to within isomorphism) H is a subdirect subgroup of $G_1 \times G_2$ satisfying $H \cap G_i = N_i$; as before, let $K = N_1 \times N_2$. Since H/K is a quotient of $H/N_i \in \mathscr{X}$, we can find $F/K \in \mathrm{Proj}^+_{\mathscr{H}}(H/K) \cap \mathscr{F}$. Condition (ii) yields that the image of F/K in G_i/N_i (with respect to the canonical projection) is some X_iN_i/N_i with $X_i \in \mathrm{Proj}^+_{\mathscr{H}}(G_i) \cap \mathscr{F}$. Now the Theorem applies (with $H_iK = F$), whence (i) holds for H.

In order to prove condition (ii), let $N \trianglelefteq H$, $E/N \in \mathrm{Proj}^+_{\mathscr{H}}(H/N)I \cap \mathscr{F}$. If $N_1 \times N_2 \leqslant N$, then our Theorem can be applied, with $E \leqslant E/N_2 \times E/N_1$, $N_1 \times N_2$, and $F(N_1 \times N_2)$ for a suitable $F \in \mathrm{Proj}^+_{\mathscr{H}}(E)$ such that $F(N_1 \times N_2)/(N_1 \times N_2) \in \mathscr{F}$ instead of $H \leqslant G_1 \times G_2$; K, and $H_1K = H_2K$, and gives the desired conclusion; note that (due to $H/N_i \in \mathscr{X}$) $X_i \in \mathrm{Proj}^+_{\mathscr{H}}(E/N_i) \cap \mathscr{F}$ can be found. If $N_i \nleqslant N$ for some $i \in \{1, 2\}$, a similar argument applies to $E \leqslant E/N \times E/E \cap N_i$. □

The preceding result gives rise to a method for showing that the formations generated by certain groups G are contained in a given homomorph $\mathscr{Y}$: one might

try to find a Schunck class $\mathscr{H}$ and a formation $\mathscr{F}$ such that $G \in X^*(\mathscr{H}, \mathscr{F})$ and $X^*(\mathscr{H}, \mathscr{F}) \cap b(\mathscr{Y}) \subseteq \mathscr{H} \backslash \mathscr{F}$; verification of condition (ii) is sometimes easy by means of the conjugacy criterion [**1**, 3.4c] for elements of $\mathrm{Proj}^+_{\mathscr{H}}(G)$. For a specific application of this type the reader is referred to [**2**, 4.1].

References

1. P. Förster, *Projektive Klassen endlicher Gruppen*. I. *Schunck- und Gaschützklassen*, Math. Z. **186** (1984), 149–178.

2. P. Förster and E. Salomon, *Local definitions of local homorphs and formations of finite groups*, Bull. Austral. Math. Soc. **31** (1985), 5–34.

3. M. P. Hale, *Pull-backs of formation projectors*, Proc. Amer. Math. Soc. **60** (1976), 61–64.

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