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# A COMMON FIXED-POINT THEOREM IN REFLEXIVE LOCALLY UNIFORMLY CONVEX BANACH SPACES[1]

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ABSTRACT. Let $X$ be a reflexive locally uniformly convex Banach space and $G$ an ultimately nonexpansive commutative semigroup of continuous self-maps of $X$. If there exists a point $x$ in $X$ recurrent under $G$ such that $G(x)$ is bounded, then $G$ has a common fixed point in $\overline{\mathrm{co}}(G(x))$. If $X$ is a Hilbert space then there is exactly one such point in $\overline{\mathrm{co}}(G(x))$.

**1. Introduction.** Let $(X, d)$ be a metric space and $G$ a family of mappings $g: X \to X$ forming a semigroup under composition. The notion of a $G$-closure point $x$ was introduced in **[5]** and defined by the condition: for some $z \in X$, any $\varepsilon > 0$, and any $f \in G$ there is a $g \in G$ such that

$$d(fg(z), x) < \varepsilon. \tag{I}$$

In **[4]** we discussed fixed point properties of semigroups, termed *ultimately nonexpansive* and defined by the condition that for every $u, v \in X$ and every $\alpha > 0$ there is an $f \in G$ such that, for all $g \in G$,

$$d(fg(u), fg(v)) \leqslant (1 + \alpha) d(u, v). \tag{II}$$

Among other things it was shown there that if $X$ is a reflexive locally uniformly convex Banach space and $G$ is an ultimately nonexpansive commutative semigroup of continuous mappings $g: X \to X$, then the existence of a point $x$ with a precompact orbit $G(x) = \{g(x): g \in G\}$ guarantees a common fixed point.

It is the purpose of this paper to prove the stronger result, obtained by replacing the hypothesis of precompactness by the assumption that there exist a $G$-closure point whose orbit is bounded. The special case where $G$ is generated by a single map $f$ was treated in **[3]**, where it was shown that the generator $f$ has a unique fixed point in $\overline{\mathrm{co}}\{f^n(x): n = 1, 2, \ldots\}$. The case of a general semigroup $G$, which is the object of this paper, is of added interest, as the $G$-closure property is, in general, weaker than the corresponding one for a single map $f$. This fact is amply reflected in the more elaborate arguments of Lemmas 1–5, which pave the way to the proof that the restriction of $G$ to $\overline{\mathrm{co}}(G(x))$ is an affine isometry (cf. §2). There seems to be no

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compelling reason to believe that the uniqueness part of [**3**] is valid in general, although it does hold in the case where $X$ is a Hilbert space.

To simplify the presentation of our main result, we introduce the notion of $G$-recurrence. Thus, a point $x \in X$ is said to be $G$-recurrent, or recurrent under $G$, if for any $\varepsilon > 0$ and any $f \in G$ there is an $h \in G$ such that

$$d(fh(x), x) < \varepsilon. \tag{III}$$

In [**4**, Proposition 1(a)] we pointed out that if $G$ is an ultimately nonexpansive commutative semigroup on any metric space, then $x$ is $G$-recurrent if it is a $G$-closure point. Clearly then, for semigroups such as those in this paper, the two notions are equivalent.

**2. Preliminaries.**

LEMMA 1. *Let $G$ be an ultimately nonexpansive commutative semigroup of continuous mappings of a Banach space $X$ into itself. Let $z, u_1, u_2, \ldots, u_n$ be members of $X$. Then to any positive integer $k$ there is a $g_k$ in $G$ with the property that, for any $g$ in $G$ and each $i = 1, 2, \ldots, n$,*

$$\|gg_k(u_i) - gg_k(z)\| \leqslant (1 + 1/k)\|u_i - z\|. \tag{1}$$

PROOF. For a fixed $i$ there exists a $g_k^{(i)}$ in $G$ such that (1) is satisfied, with $g_k^{(i)}$ replacing $g_k$. Clearly then, (1) holds with $g_k = g_k^{(1)} g_k^{(2)} \cdots g_k^{(n)}$.

LEMMA 2. *Let $z_1, z_2 \in X$, $g \in G$, where $G$ is as in Lemma 1, and suppose that a sequence $\{g_k\} \subset G$ exists such that, for all $h \in G$,*

$$\|hg_k g(z_1) - hg_k g(z_2)\| \leqslant (1 + 1/k)\|z_1 - z_2\| \tag{2}$$

*and*

$$\|hg_k g(z_1) - hg_k g(z_2)\| \leqslant (1 + 1/k)\|g(z_1) - g(z_2)\| \tag{3}$$

*for $k = 1, 2 \ldots.$*

*Suppose further that sequences $\{h_k\}$, $\{h'_k\} \subset G$ exist such that $\lim_{k \to \infty} h'_k g_k g(z_i) = g(z_i)$ and $\lim_{k \to \infty} h_k g_k g(z_i) = z_i$ $(i = 1, 2)$. Then*

$$\|g(z_1) - g(z_2)\| = \|z_1 - z_2\|.$$

PROOF. Substituting $h'_k$ and $h_k$ for $h$ in (2) and (3), respectively, we obtain two inequalities from which the result follows. Indeed,

$$\|g(z_1) - g(z_2)\| = \lim_{k \to \infty} \|h'_k g_k g(z_1) - h'_k g_k g(z_2)\| \leqslant \left(1 + \frac{1}{k}\right)\|z_1 - z_2\|$$

and

$$\|z_1 - z_2\| = \lim_{k \to \infty} \|h_k g_k g(z_1) - h_k g_k g(z_2)\| \leqslant \left(1 + \frac{1}{k}\right)\|g(z_1) - g(z_2)\|,$$

whence, since $k = 1, 2, \ldots$ is arbitrary, $\|g(z_1) - g(z_2)\| \leqslant \|z_1 - z_2\|$ and, simultaneously, $\|z_1 - z_1\| \leqslant \|g(z_1) - g(z_2)\|$.

LEMMA 3. *Let $X$ be a reflexive locally uniformly convex Banach space, and let $G$ be as in Lemma 1. Suppose that $p, q, z \in X$ and $\{g_k\} \subset G$ are such that*

$$z = \lambda p + (1 - \lambda) q$$

*for some $\lambda$, $0 < \lambda < 1$, and*

$$\begin{aligned} \|gg_k(p) - gg_k(z)\| &\leqslant (1 + 1/k)\|p - z\|, \\ \|gg_k(q) - gg_k(z)\| &\leqslant (1 + 1/k)\|q - z\| \end{aligned} \tag{5}$$

*for all $g \in G$.*

*Suppose further that a sequence $\{h_k\} \subset G$ exists such that $\{h_k g_k(p)\}$ and $\{h_k g_k(q)\}$ both converge and $\lim_{k\to\infty} h_k g_k(p) = p$, $\lim_{k\to\infty} h_k g_k(q) = q$. Then $\{h_k g_k(z)\}$ converges, and $\lim_{k\to\infty} h_k g_k(z) = z$.*

PROOF. In (5) we may replace $g$ by members of the sequence $\{h_k\}$ and observe that, since the sequences $\{h_k g_k(p)\}$ and $\{h_k g_k(q)\}$ are bounded, so is $\{h_k g_k(z)\}$. By reflexivity of $X$ some subsequence $\{h_{k_j} g_{k_j}(z)\}$ converges weakly to, say, $w \in X$. Since norms are weakly lower semicontinuous, it follows from the above inequalities that $\|p - w\| \leqslant \|p - z\|$ and $\|q - w\| \leqslant \|q - z\|$. Hence,

$$\|p - q\| = \|p - z\| + \|q - z\| \geqslant \|p - w\| + \|q - w\| \geqslant \|p - q\|,$$

clearly implying that $\|p - w\| = \|p - z\|$ and $\|q - w\| = \|q - z\|$. Hence, $w = z$ by strict convexity of $X$; and because this is true for each weakly convergent subsequence of $\{h_k g_k(z)\}$, it is also true that the entire sequence converges weakly to $z$.

Now the vectors

$$\left[(1 - 1/k)\|p - z\|\right]^{-1}\left(h_k g_k(p) - h_k g_k(z)\right) \qquad (k = 1, 2, \ldots)$$

are all of norm $\leqslant 1$ and form a sequence which converges weakly to

$$(p - z)\left[\|p - z\|\right]^{-1}$$

on the unit sphere. By a known property of locally uniformly convex Banach spaces (cf. [**1**, p. 32]), the same sequence converges in norm. Hence,

$$\lim_{k\to\infty}\left(h_k g_k(p) - h_k g_k(z)\right) = p - z \quad \text{and} \quad \lim_{k\to\infty} h_k g_k(z) = z,$$

as claimed.

LEMMA 4. *Let $X$ and $G$ be as in Lemma 2. Suppose $x \in X$ is recurrent under $G$, and let $u_1, u_2 \in G(x)$. Then the restriction of each member $g$ of $G$ to the line segment $[u_1, u_2]$ is an affine isometry.*

PROOF. Let $z_1$, $z_2$ be points on the line segment $[u_1, u_2]$. Since all isometries in a strictly convex Banach space are affine, it suffices to show that $\|g(z_1) - g(z_2)\| = \|z_1 - z_2\|$. To this end let $\{g_k\} \subset G$ be a sequence with the property that

$$\|hg_k g(u) - hg_k g(v)\| \leqslant (1 + 1/k)\|u - v\|$$

and

$$\|hg_k g(u) - hg_k g(v)\| \leqslant (1 + 1/k)\|g(u) - g(v)\|$$

for all $h \in G$, all $k = 1, 2, \ldots$, and all $u, v$ in the set $\{u_1, u_2, z_1, z_2, g(u_1), g(u_2), g(z_1), g(z_2)\}$. Let $\{h_k\}$ be a sequence in $G$ with the property that $\lim_{k\to\infty} h_k g_k g(x) = x$. Then, by the continuity of members of $G$, $\lim_{k\to\infty} h_k g_k g(u_i) = u_i$, $i = 1, 2$. By Lemma 3, $\lim_{k\to\infty} h_k g_k g(z_i) = z_i$, $i = 1, 2$. Next, set $h'_k = h_k g$. Then

$$\lim_{k\to\infty} h'_k g_k g(u_i) = g(u_i) \quad \text{and} \quad \lim h'_k g_k g(z_i) = g(z_i)$$

for $i = 1, 2$. Lemma 2 applies to the effect that $\|g(z_1) - g(z_2)\| = \|z_1 - z_2\|$.

LEMMA 5. *Let $X$ be a reflexive locally uniformly convex Banach space and $G$ an ultimately nonexpansive commutative semigroup of continuous mappings of $X$ into itself. If $x \in X$ is recurrent under $G$, then the restriction of each $g \in G$ to $\overline{\mathrm{co}}(G(x))$ is an affine isometry.*

PROOF. Let $n \geqslant 2$ be a positive integer and suppose that $z_1, z_2 \in \mathrm{co}\{u_1, u_2, \ldots, u_n\}$, $z_1 \neq z_2$. For $n = 2$, $\|g(z_1) - g(z_2)\| = \|z_1 - z_2\|$ by Lemma 4. Suppose this is true for $z_1, z_2 \in \mathrm{co}\{u_1, u_2, \ldots, u_m\}$ with $m \leqslant n - 1$. Let $p_1, p_2$ be extreme points of the line segment $l \cap \mathrm{co}\{u_1, u_2, \ldots, u_n\}$, where $l$ is the straight line through $z_1$ and $z_2$. Choose $\{g_k\} \subset G$ so as to satisfy the two inequalities of Lemma 2. Further, let $\{h_k\}$ and $\{h'_k\}$ be as in the proof of Lemma 4; that is, $\lim_{k\to\infty} h_k g_k g(x) = x$ and $\lim h'_k g_k g(x) = g(x)$. Now $p_1, p_2$ are each convex combinations of $m$ of the points of $\{u_1, u_2, \ldots, u_n\}$, with $m \leqslant n - 1$, and $g$ is affine on the convex hull of such sets. Suppose $p_1 = \sum_{i=1}^m \lambda'_i u_i$ and $p_2 = \sum_{i=1}^m \lambda''_i u_i$ for suitable $\lambda'_i$, $\lambda''_i$, with $0 \leqslant \lambda'_i, \lambda''_i \leqslant 1$ and $\sum_{i=1}^m \lambda'_i = 1 = \sum_{i=1}^m \lambda''_i$. We then obtain

$$g(p_1) = \sum_{i=1}^m \lambda'_i g(u_i) \quad \text{and} \quad g(p_2) = \sum_{i=1}^m \lambda''_i g(u_i).$$

Hence,

$$\lim_{k\to\infty} h_k g_k g(p_1) = \sum_{i=1}^m \lim_{k\to\infty} \lambda'_i h_k g_k g(u_i) = \sum_{i=1}^m \lambda'_i u_i = p_1,$$

and, similarly, $\lim h'_k g_k g(p_1) = g(p_1)$; likewise,

$$\lim_{k\to\infty} h_k g_k g(p_1) = p_2 \quad \text{and} \quad \lim_{k\to\infty} h'_k g_k g(p_2) = g(p_2).$$

By Lemma 3 the above equations remain valid with $z_1$, $z_2$ replacing $p_1$, $p_2$. By Lemma 2, $\|g(z_1) - g(z_2)\| = \|z_1 - z_2\|$, and, again by strict convexity of $X$, $g$ is affine. Hence, $g$ is affine on $\mathrm{co}(G(x))$ and, by continuity, $g$ is also affine on $\overline{\mathrm{co}}(G(x))$.

THEOREM. *Let $X$ be a reflexive locally uniformly convex Banach space and $G$ an ultimately nonexpansive commutative semigroup of continuous self-maps of $X$. If an $x \in X$ exists such that $G(x)$ is bounded and $x$ is a recurrent point under $G$, then $\overline{\mathrm{co}}\, G(x)$ contains a point $\xi$ such that $G(\xi) = \{\xi\}$. If, in addition, $X$ is a Hilbert space, then $\xi$ is unique with the above property; i.e., if $\eta \neq \xi$ belongs to $\overline{\mathrm{co}}\, G(x)$ then $g(\eta) \neq \eta$ for some $g \in G$.*

PROOF. By Lemma 5, $G|\overline{\mathrm{co}}\, G(x)$, the semigroup consisting of restrictions of members of $G$ to $\overline{\mathrm{co}}\, G(x)$, is composed of affine isometries. By the Markov-Kakutani Theorem **[2]** there exists a common fixed point. To prove the assertion about uniqueness, assume $\eta \neq \xi$ is another common fixed point in $\overline{\mathrm{co}}\, G(x)$ and let $l$ be the straight line joining $\xi$ and $\eta$. Let $\bar{g}$ be the affine isometry on the affine hull of $l \cap \{\overline{\mathrm{co}}\, G(x)\}$, which is determined by $g \in G$. Then, for every $\alpha \in l$, $\|\bar{g}(x) - \bar{g}(\alpha)\| = \|x - \alpha\|$. In particular, $\|\bar{g}(x) - \bar{g}(\zeta)\| = \|x - \zeta\|$, where $\zeta$ is the point of $l$ nearest to $x$. Because $x - \zeta$ is perpendicular to $l$ in an inner product space, we have $\langle x - \zeta, \xi - \eta\rangle = 0$, and because all distances are preserved under $\bar{g}$ and $l$ is pointwise fixed, $\zeta$ is also the nearest point of $l$ to $\bar{g}(x) = g(x)$. It follows that $\langle g(x) - \zeta, \xi - \eta\rangle = 0$ for all $g \in G$ and, as an easy consequence, $l \cap \{\overline{\mathrm{co}}(G(x))\}$ is a singleton. Thus, $\overline{\mathrm{co}}\, G(x)$ cannot contain $\{\xi, \eta\}$, and the proof is complete.

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