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ARENS PRODUCT AND THE ALGEBRA OF DOUBLE MULTIPLIERS

PAK-KEN WONG

ABSTRACT. Let A be a semisimple Banach algebra and $M(A)$ the algebra of double multipliers of A. We show that $M(A)$ is isomorphic to $(A^{**}, \circ)$ if and only if A has the following properties: (1) A is Arens regular, (2) A has a weak approximate identity, and (3) $\pi(A)$ is an ideal of $(A^{**}, \circ)$.

1. Notation and preliminaries. Definitions not explicitly given in this paper are taken from Rickart's book [**6**].

Let A be a Banach algebra, let A^* and A^{**} be the conjugate and second conjugate spaces of A, respectively, and let π be the canonical embedding of A into A^{**}. There are two Arens products $\circ$ and $\circ'$ defined on A^{**} (e.g. see [**3** and **10**]). Under either product $\circ$ or $\circ'$, A^{**} becomes a Banach algebra. In general, $\circ$ and $\circ'$ are distinct on A^{**}. If they coincide on A^{**}, then A is called *Arens regular*. The left (resp. right) multiplication is weakly continuous in $(A^{**}, \circ)$ (resp. $(A^{**}, \circ')$). If $x \in A$ and $F \in A^{**}$, then $\pi(x) \circ F = \pi(x) \circ' F$ and $F \circ \pi(x) = F \circ' \pi(x)$.

We say that a Banach algebra A has a *weak approximate identity* if there exists a net $\{e_\alpha\}$ in A such that $\{e_\alpha\}$ is bounded and $\lim f(e_\alpha x - x) = \lim f(x e_\alpha - x) = 0$ for all $x \in A$ and $f \in A^*$.

Let A be a semisimple Banach algebra. A pair (T_1, T_2) of operators from A to A is called a *double multiplier* (*double centralizer*) on A provided that $x(T_1 y) = (T_2 x)y$ for all x, y in A. It is known that T_1 and T_2 are continuous linear operators on A such that $T_1(xy) = (T_1 x)y$ and $T_2(xy) = x(T_2 y)$ for all x, y in A. The set $M(A)$ of all double multipliers on A is a Banach algebra with identity and A can be identified as an ideal of $M(A)$ (see [**2** and **4**]).

In this paper, all algebras and spaces under consideration are over the complex field.

2. The main result. If A is a B^*-algebra, then A is Arens regular and A has an approximate identity. Also, $M(A) = A^{**}$ if and only if $\pi(A)$ is an ideal of $(A^{**}, \circ)$ (see [**8**, Theorem 2.2, p. 80]). For a different proof of this result, see [**1**, Theorem 2.8, p. 282].

We have the following generalization of the above result.

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THEOREM. *Let A be a semisimple Banach algebra. Then $M(A)$ is isomorphic to $(A^{**}, \circ)$ if and only if A has the following properties*:

(1) *A is Arens regular.*

(2) *A has a weak approximate identity $\{e_\alpha\}$.*

(3) *$\pi(A)$ is an ideal of $(A^{**}, \circ)$.*

PROOF. Assume $M(A)$ is isomorphic to $(A^{**}, \circ)$. Since A is an ideal of $M(A)$, $\pi(A)$ is an ideal of $(A^{**}, \circ)$. Let $F, G, H \in A^{**}$ and let $\{x_t\}$ be a net in A such that $\pi(x_t) \to H$ weakly. Since $\pi(A)$ is an ideal of $(A^{**}, \circ)$, it follows that

$$\pi(x_t)\circ(F\circ G) = (\pi(x_t)\circ F)\circ G = (\pi(x_t)\circ F)\circ' G = (\pi(x_t)\circ' F)\circ' G$$
$$= \pi(x_t)\circ'(F\circ' G) = \pi(x_t)\circ(F\circ' G).$$

Therefore $H\circ(F\circ G) = H\circ(F\circ' G)$ for all H in A^{**}. Since $(A^{**}, \circ)$ is isomorphic to $M(A)$, $(A^{**}, \circ)$ has an identity I. Put $H = I$, then we have $F\circ G = F\circ' G$. Therefore A is Arens regular. Since $(A^{**}, \circ)$ has an identity and $\circ$ coincides with $\circ'$, it follows from the proof of [**3**, Lemma 3.8, p. 855] that A has a weak approximate identity. Therefore A has properties (1), (2) and (3).

Conversely, assume that A has properties (1), (2) and (3). Then it follows from the proof of [**3**, Lemma 3.8, p. 855] that $(A^{**}, \circ)$ has an identity I. Let $T = (T_1, T_2) \in M(A)$. For each $f \in A^*$, define

$$(f\circ T_1)(x) = f(T_1 x) \quad \text{and} \quad (f\circ T_2)(x) = f(T_2 x) \qquad (x \in A).$$

Then $f\circ T_1$ and $f\circ T_2 \in A^*$. Since $\{\pi(T_1 e_\alpha)\}$ is bounded, by Alaoglu's Theorem, it has a weak limit point T_1' in A^{**}. Assume that $\pi(T_1 e_{\alpha_1}) \to T_1'$ weakly in A^{**}. For any x in A and f in A^*, we have

$$\begin{aligned}(T_1'\circ\pi(x))(f) &= T_1'(\pi(x)\circ f) = \lim \pi(T_1 e_{\alpha_1})(\pi(x)\circ f)\\ &= \lim \pi((T_1 e_{\alpha_1})x)(f) = \lim f(T_1(e_{\alpha_1}x))\\ &= \lim(f\circ T_1)(e_{\alpha_1}x) = (f\circ T_1)(x)\\ &= f(T_1 x) = \pi(T_1 x)(f).\end{aligned}$$

Hence $T_1'\circ\pi(x) = \pi(T_1 x)$. Assume that $\{\pi(T_1 e_\alpha)\}$ has another weak limit point T_1''. Then

$$T_1'\circ\pi(x) - T_1''\circ\pi(x) = \pi(T_1 x) - \pi(T_1 x) = 0.$$

Since, by Goldstine's Theorem, $\pi(A)$ is weakly dense in A^{**}, it follows that $(T_1' - T_1'')\circ A^{**} = (0)$. Since A^{**} has an identity element I, it follows that $T_1' = T_1''$. Consequently, T_1' is unique. Similarly, $\{\pi(T_2 e_\alpha)\}$ has a unique weak limit point T_2' such that $\pi(x)\circ T_2' = \pi(T_2 x)$ for all x in A. Since $x(T_1 y) = (T_2 x)y$ for all x, y in A, it follows that $\pi(x)\circ T_1' = \pi(T_2 x)$ and $T_2'\circ\pi(x) = \pi(T_1 x)$. Now let

$$T' = \tfrac{1}{2}(T_1' + T_2') \qquad (T = (T_1, T_2) \in M(A)).$$

It is easy to see that $\pi(x)\circ T' = \pi(T_2 x)$ and $T'\circ\pi(x) = \pi(T_1 x)$. We show that $T \to T'$ is an isomorphism of $M(A)$ onto A^{**}. In fact, let $S = (S_1, S_2)$ and $T = (T_1, T_2) \in M(A)$. Since $ST = (S_1 T_1, T_2 S_2)$, we have

$$\pi(x)(S'\circ T') = \pi(S_2 x)\circ T' = \pi(T_2 S_2 x) = \pi(x)\circ(ST)'$$

for all x in A. Hence $A^{**} \circ ((ST)' - S' \circ T') = (0)$ and so $(ST)' = S' \circ T'$. If $T' = 0$, then $\pi(T_2 x) = \pi(x) \circ T' = 0$ for all x in A. Hence $T_2 = 0$. Similarly, $T_1 = 0$ and therefore $T = (T_1, T_2) = 0$. Let $F \in A^{**}$. Define

$$P_1(x) = F \circ \pi(x) \quad \text{and} \quad P_2(x) = \pi(x) \circ F \qquad (x \in A).$$

Since $\pi(A)$ is an ideal of A^{**}, it follows that $P = (P_1, P_2) \in M(A)$ and $P' = F$. Therefore the mapping $T \to T'$ is an isomorphism from $M(A)$ onto A^{**}. It is easy to see that this isomorphism is continuous. This completes the proof of the theorem.

A part of the proof of the above theorem is similar to that given in [**8**, Lemma 2.1, p. 80].

Let A be the group algebra of a compact group. Then $\pi(A)$ is an ideal of $(A^{**}, \circ)$ [**8**, Corollary 3.4, p. 83] and A has an approximate identity. Since $(A^{**}, \circ)$ does not have an identity, $M(A)$ cannot be isomorphic to $(A^{**}, \circ)$. Therefore A is not Arens regular.

In conclusion, we give a remark about the condition "$\pi(A)$ is an ideal of $(A^{**}, \circ)$".

An element x in a Banach algebra A is said to be weakly completely continuous (w.c.c.) if the left and right multiplication operators of x are weakly completely continuous. If each element of A is w.c.c., then A is called w.c.c.

LEMMA. *Let A be a Banach algebra. Then $\pi(A)$ is an ideal of $(A^{**}, \circ)$ if and only if A is w.c.c.*

PROOF. Suppose that $\pi(A)$ is an ideal of A^{**}. Let $\{x_\alpha\}$ be a bounded net in A. Then by Alaoglu's Theorem, we can assume that $\pi(x_\alpha) \to F$ weakly for some F in A^{**}. Since $\pi(x) \circ F \in \pi(A)$ for all x in A, it follows that $\pi(x x_\alpha) \to \pi(x) \circ F$ weakly in $\pi(A)$. Hence A is w.c.c.

Conversely, suppose that A is w.c.c. Let $x \in A$ and $F \in A^{**}$ with $\|F\| = 1$. By Goldstine's Theorem, there exists a net $\{x_\alpha\} \subset A$ with $\|x_\alpha\| \leq 1$ such that $\pi(x_\alpha) \to F$ weakly. Since x is w.c.c., we can assume that $x x_\alpha \to y$ weakly for some y in A. It follows that $\pi(x) \circ F = \pi(y) \in \pi(A)$. Consequently $\pi(A)$ is an ideal of A^{**} and this completes the proof.

Let A be a B^*-algebra. Then A is a dual algebra if and only if $\pi(A)$ is an ideal of $(A^{**}, \circ)$ [7, Theorem 5.1, p. 533]. This result actually follows easily from [**5**, Theorem 8, p. 24] (which states that A is a dual algebra if and only if A is w.c.c.). For generalizations of this result, see [**9**, Theorem 5.2, p. 830 and **10**, Theorem 3.1, p. 439].

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DEPARTMENT OF MATHEMATICS, SETON HALL UNIVERSITY, SOUTH ORANGE, NEW JERSEY 07079