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ON A THEOREM OF COHEN AND MONTGOMERY

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ABSTRACT. In a recent paper, Cohen and Montgomery proved a conjecture of Bergman concerning the relation between the Jacobson radical and the graded Jacobson radical of a ring graded by a finite group. In their proof they made use of the theory of Hopf algebras. In this note we give a short and elementary proof of the Bergman conjecture.

Let R be a G-graded ring, where G is a finite group. Let $J(R)$ and $J_g(R)$ denote the Jacobson radical and the graded Jacobson radical of R, respectively. In **[1]** Bergman conjectured that $J_g(R) \subset J(R)$. This conjecture is nicely complemented by the question of whether $J(R) = J_g(R)$ if $|G|^{-1} \in R$. Both questions are answered affirmatively by Cohen and Montgomery in **[2]**. In their paper they used the observation that R is G-graded if and only if R is a $k[G]^*$ module algebra. (In a Hopf algebra context, k is always a small commutative ring such that everything that is used has a k-structure.) In this note we use $k[G]$ comodule algebras instead of $k[G]^*$ module algebras. This allows us to give a very short proof of Bergman's conjecture. We also obtain a short proof for $J(R) = J_g(R)$ if $|G|^{-1} \in R$. For any Hopf algebra H a ring R is an H comodule algebra if there exists an algebra homomorphism $\rho\colon R \to R \otimes H$ satisfying

(1) $(\mathrm{id} \otimes \varepsilon) \circ \rho = \mathrm{id}$,

(2) $(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta) \circ \rho$.

Now it is elementary that R is G-graded $\Leftrightarrow$ R is a $k[G]$-comodule algebra, where the map ρ is given by

$$\rho\colon R \to R \otimes kG\colon r \to \sum_{x \in G} r_x \otimes x.$$

In the rest of this note we will identify $R \otimes kG$ with RG, which will be considered to be graded in the usual way. We will denote the image of R under ρ by S.

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(1) THEOREM. $J_g(R) \subset J(R)$.

PROOF. Let V be an irreducible R-module. Then $W = V^G = \sum_{\sigma \in G} \sigma V$ is clearly a graded irreducible RG-module. Since RG is finitely generated, as S-module this

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implies that W is finitely generated as S-module. The graded version of Nakayama's lemma now implies that $J_g(S)W \neq W$. Set $J_g(R) = \Sigma_{\sigma \in G} I_\sigma$. Then $J_g(S) = \Sigma_{\sigma \in G} \sigma I_\sigma$. So we see that

$$W \neq J_g(S)W = (\Sigma \sigma I_\sigma)W = (\Sigma I_\sigma)W = J_g(R)W.$$

Since $J_g(R)W$ is clearly a graded submodule of W, then $J_g(R)W = 0$, and hence, $J_g(R)V = 0$. This implies that $J_g(R) \subset J(R)$.

Now we will prove the opposite inclusion $J(R) \subset J_g(R)$ if $|G|^{-1} \in R$.

(2) LEMMA (Graded version of Maschke's theorem). *Suppose $|G|^{-1} \in R$. If M, N are two graded RG-modules, then there is a map* $\sim$: $\mathrm{Hom}_S(M, N) \to \mathrm{Hom}_{RG}(M, N)$ *satisfying the following conditions*:

(a) *The image of a map of degree zero is a map of degree zero.*

(b) *If $f \in \mathrm{Hom}_S(M, N)$ is RG-linear, then $\tilde{f} = f$.*

(c) *Suppose M' and N' are two other RG-modules. If there is a commutative diagram of RG-modules*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \alpha \downarrow & & \downarrow \beta \\ M' & \xrightarrow{g} & N' \end{array}$$

where α, β are RG-linear and f, g are S-linear, then the diagram

$$\begin{array}{ccc} M & \xrightarrow{\tilde{f}} & N \\ \alpha \downarrow & & \downarrow \beta \\ M' & \xrightarrow{\tilde{g}} & N' \end{array}$$

is also commutative.

PROOF. Let $f \in \mathrm{Hom}_S(M, N)$, $m \in M$, and define

$$\tilde{f}(m) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma f(\sigma^{-1} m).$$

The demonstration that $\tilde{f}$ is RG-linear is entirely classical. Let $\tau \in G$. Then

$$\tilde{f}(\tau m) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma f(\sigma^{-1} \tau m) = \frac{1}{|G|} \sum_{\sigma' \in G} \tau \sigma' f(\sigma'^{-1} m) = \tau \tilde{f}(m).$$

It is obvious that $\tilde{f}$ satisfies properties (a)–(c).

(3) COROLLARY. *Let V be a graded RG-module that is graded completely reducible as S-module. Then V is graded completely reducible as RG-module.*

PROOF. Again, this is entirely classical. Let W be a graded RG-submodule of V. Then the inclusion map $W \to V$ splits as a map of S-modules. Hence, by Lemma (2), $W \to V$ splits as a map of RG-modules.

If $\tau \in G$ and M is a graded S-module, then $M(\tau)$ is defined as the graded S-module obtained by putting $(M(\tau))_\mu = M_{\mu\tau}$. It is clear that the functor $M \to M(\tau)$ defines an autoequivalence on the category of graded S-modules.

(4) LEMMA. *Let M be a graded S-module. Then $RG \otimes_S M \simeq \sum_{\tau \in G} M(\tau)$ as graded S-modules.*

PROOF.

$$RG \otimes_S M = \sum_\tau \sum_\sigma \sigma\tau^{-1}\sigma^{-1} \otimes M_\sigma$$

Now it is easy to check that $\sum_\sigma \sigma\tau^{-1}\sigma^{-1} \otimes M_\sigma$ is a graded S-module. The map

$$\sum_\sigma \sigma\tau^{-1}\sigma^{-1} \otimes M_\sigma \to M(\tau) : \sum \sigma\tau^{-1}\sigma^{-1} \otimes m_\sigma \to \sum m_\sigma$$

is a graded isomorphism of S-modules.

(5) THEOREM. *If $|G|^{-1} \in R$, then $J(R) \subset J_g(R)$.*

PROOF. The proof is an imitation of the proof of Theorem 7.1 of [**3**]. Let V be a graded, completely reducible S-module. Then $RG \otimes_S V$ is clearly a graded, completely reducible S-module, and, hence, ${}_{RG}(RG \otimes_S V)$ is graded, completely reducible. In particular, $J_g(RG)(RG \otimes_S V) = 0$. Let $\alpha \in J_g(RG)$. Then α may be written as $\sum_{g \in G} g \cdot s^{(g)}$. Then for any $v \in V$, $0 = \alpha(1 \otimes v) = \sum g \otimes s^{(g)} v$. So $s^{(g)} V = 0$. Since this is true for all such V, we deduce that $s^{(g)} \in J_g(S)$, and, therefore, $J_g(S)G \supset J_g(RG)$. Taking parts of degree zero yields $J(R) \subset J_g(R)$.

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