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ON THE WELL-POSEDNESS OF A C^∞ GOURSAT PROBLEM FOR A PARTIAL DIFFERENTIAL OPERATOR OF ORDER GREATER THAN TWO

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ABSTRACT. We find a necessary and sufficient condition for a Goursat problem for a third order partial differential operator with constant coefficients of the form

$$C_2(D_x, D_y)D_t + C_3(D_x, D_y)$$

to be C^∞-well posed, showing at the same time that a necessary and sufficient condition of Hasegawa cannot be extended. The result can be generalised to operators of higher orders but leads to cumbersome conditions; nevertheless, we show that the condition of Hasegawa is also not sufficient in this case.

Let A be a partial differential operator of degree m with real constant coefficients, and $A(\xi)$ be the corresponding polynomial; we write $A_j(\xi)$ for the homogeneous part of degree j of A, $A_m(\xi)$ being the principal part.

In [3], Nishitani studied the following Goursat problem:

$$\text{(P)}\quad \begin{cases} A(D_t, D_x, D_y)u = 0, & t > 0, x \in \mathbf{R}, y \in \mathbf{R}^n, \\ D_x^k u = g_k(t, y), & t > 0, y \in \mathbf{R}^n, x = 0, k = 0, 1, \ldots, l-1, \\ D_t^j u = h_j(x, y), & t = 0, x \in \mathbf{R}, y \in \mathbf{R}^n, j = 0, 1, \ldots, m-l-1, \end{cases}$$

with the compatibility conditions

$$D_x^k h_j(0, y) = D_t^j g_k(0, y), \qquad k = 0, 1, \ldots, l-1, j = 0, 1, \ldots, m-l-1, y \in \mathbf{R}^n,$$

where A is written as follows:

$$A(\tau, \zeta, \eta) = \sum_{j=l}^{m} C_j(\zeta, \eta)\tau^{m-j}$$

where $C_j(\zeta, \eta)$ is a polynomial of degree at most j, and $C_l^0(1, 0) = 1$ where $C_l^0(\zeta, \eta)$ is the principal part of C_l.

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Nishitani proved

THEOREM 1. *If the problem* (P) *is C^∞-well posed* (*that is, there exists a unique solution $u \in C^\infty(\overline{\mathbf{R}}_+ \times \mathbf{R} \times \mathbf{R}^n)$ of* (P)) *for every $g_k \in C^\infty(\overline{\mathbf{R}}_+ \times \mathbf{R}^n)$ and $h_j \in C^\infty(\mathbf{R} \times \mathbf{R}^n)$ satisfying the compatibility conditions, then A is written as follows*:

$$A(\tau, \zeta, \eta) = \hat{C}_l(\zeta, \eta)\hat{Q}_{m-l}(\tau, \zeta, \eta) + R_{m-2}(\tau, \zeta, \eta) \tag{1}$$

where $\hat{C}_l$ and $\hat{Q}_{m-l}$ are polynomials with principal part C_l^0 and Q_{m-l} respectively, Q_{m-l} is a homogeneous polynomial of degree $m - l$ hyperbolic with respect to $(1,0,0)$, *and R_{m-2} is a polynomial of degree at most $m - 2$.*

Hasegawa [1] showed that if $l = 1$ and Q_{m-1} is strictly hyperbolic with respect to $(1,0,0)$, then the condition of Theorem 1 is sufficient.

We will show that for $m \geqslant 3$ arbitrary and $l = m - 1$, this condition is not sufficient (we consider $n = 1$, the case n arbitrary being similar). In the case $m = 3$, we will even obtain a necessary and sufficient condition.

A. We begin by studying the case $m = 3$. By Theorem 1, in this case we can consider

$$A(\tau, \zeta, \eta) = \hat{C}_2(\zeta, \eta)\hat{Q}_1(\tau, \zeta, \eta) + R_1(\tau, \zeta, \eta)$$

where

$$\begin{aligned} \hat{C}_2(\zeta, \eta) &= \zeta^2 + a\eta^2 + b\zeta\eta + c\zeta + d\eta + e, \\ \hat{Q}_1(\tau, \zeta, \eta) &= \tau + f\zeta + g\eta + h, \\ R_1(\tau, \zeta, \eta) &= j\tau + p\zeta + q\eta + r. \end{aligned}$$

We will prove

THEOREM 2. *The problem* (P) *is C^∞-well posed for $l = 2$ and $m = 3$ if and only if A is decomposed as in* (1) *and*

$$\text{(H)} \qquad \begin{cases} 4a - b^2 < 0 \quad \text{or} \\ \begin{cases} 4a = b^2, 2d = bc, \quad \text{and} \\ 2g = bf \Rightarrow bp = 2q, \\ a = b = 0 \Rightarrow gj = q, \\ g = 0 \Rightarrow bfj + 2q = bq. \end{cases} \end{cases}$$

In the proof of this theorem we will use the following theorem of Nishitani [3] several times:

THEOREM 3. *The problem* (P) *is C^∞-well posed if and only if there exists a positive constant ε such that, for every δ with $0 < |\delta| \leqslant \varepsilon$, the polynomial $A(\tau, \zeta, \eta)$ is hyperbolic with respect to $N = (1, \delta, 0)$.*

To begin we prove a lemma:

LEMMA 1. *A_3 is hyperbolic with respect to N if and only if $4a - b^2 \leqslant 0$.*

PROOF. By definition, A_3 is hyperbolic with respect to N if and only if $A_3(N) \neq 0$ and we can find $t_0 \in \mathbf{R}$ such that, for all $\xi \in \mathbf{R}^3$,

$$A_3(\xi + itN) = 0 \Rightarrow t \geqslant t_0.$$

We have

$$A_3(\xi) = (\zeta^2 + a\eta^2 + b\zeta n)(\tau + f\zeta + g\eta)$$

and so we can see that the only possibly unbounded roots in t of $A_3(\xi + itN) = 0$ are those given by $t^2 = 4\eta^2(4a - b^2)/\delta^2$ and the conclusion follows.

We say that the polynomial Q is weaker than the polynomial P, and write $Q \prec P$ if there exists a constant C such that, for all $\xi \in \mathbf{R}^3$, $|Q(\xi)| \leqslant C\tilde{P}(\xi)$ where

$$\tilde{P}(\xi) = \left(\sum_{|\alpha| \leqslant m} |D^\alpha P(\xi)|^2 \right)^{1/2}$$

where m is the degree of P.

LEMMA 2. *Let us suppose that A_3 is hyperbolic with respect to N. Then $A \prec A_3$ if and only if we have* (H).

PROOF. If $A \prec A_3$ (Hörmander, [**2**, p. 135]), then $A_j \prec A_3$, $j = 0, 1, 2, 3$, and so $A_0 + A_1 \prec A_3$ and $A_2 + A_3 \prec A_3$ (Hörmander, [**2**, p. 71]). These two conditions are equivalent to $A \prec A_3$. But if A_3 is hyperbolic with respect to N, then by a theorem of Svensson [**4**, p. 151], the fact that $A_2 + A_3 \prec A_3$ is equivalent to the fact that $A_2 + A_3$ is hyperbolic with respect to N. In nearly the same way as in the previous lemma, we see that this is equivalent to

$$(\mathrm{H}_1) \qquad \begin{cases} 4a - b^2 < 0 & \text{or} \\ 4a = b^2 \quad \text{and} \quad 2d = bc. \end{cases}$$

Some calculations show that the conditions $A_0 + A_1 \prec A_3$ and (H_1) are equivalent to (H).

PROOF OF THEOREM 2. If (P) is C^∞-well posed then A is decomposed as in (1) by Theorem 1, and by Theorem 3 A is hyperbolic with respect to N. So $A \prec A_3$ (because if A is hyperbolic with respect to N, then A_3 is also hyperbolic with respect to N), and by Lemma 2 we have (H).

If A is decomposed as in (1) and we have (H), it follows from Lemma 1 that A_3 is hyperbolic with respect to N. Using Lemma 2 we can now conclude that $A \prec A_3$ and, by the theorem of Svensson already cited, we have that A is hyperbolic with respect to N, and so, by Theorem 3, the problem (P) is C^∞-well posed. Q.E.D.

COROLLARY. *If A is homogeneous, the problem is C^∞-well posed if and only if*

$$A(\tau, \zeta, \eta) = (\zeta^2 + a\eta^2 + b\zeta\eta)(\tau + f\zeta + g\eta)$$

and $4a - b^2 \leqslant 0$.

REMARKS. (1) In this case $\hat{Q}_1$ is always strictly hyperbolic with respect to $(1, 0, 0)$, showing that a result similar to the one of Hasegawa is not valid for $m = 3$, $l = 2$.

(2) For the case $y \in \mathbf{R}^n$ we have a condition similar to (H), complicated by the existence of the coefficients of $D_{y_j} D_{y_k}$ in $\hat{C}_2$.

(3) Theorem 2 is easily generalised to the case of complex coefficients.

B. The method used to handle the case $m = 3$, $l = 2$ can be used to obtain a necessary and sufficient condition for each $m \geqslant 4$, $l = m - 1$, but, as we can see looking at Lemma 2, the conditions corresponding to (H) would be very cumbersome ones.

We can show, however, that the necessary and sufficient condition of Hasegawa is not sufficient for any $m \geqslant 4$, $l = m - 1$.

By Theorem 1, the principal part of A will have the form

$$A_m(\xi) = \left(\zeta^{m-1} + \sum_{k=1}^{m-1} a_k \zeta^{m-1-k}\eta^k\right)(\tau + b\zeta + c\eta)$$

with $\xi = (\tau, \zeta, \eta)$. We remark that $\hat{Q}_1(\xi) = \tau + b\zeta + c\eta + d$ is strictly hyperbolic with respect to $(1,0,0)$. The homogeneous part of degree $m - 1$ of A will have the form

$$\begin{aligned} A_{m-1}(\xi) = {} & d\left(\zeta^{m-1} + \sum_{k=1}^{m-1} a_k \zeta^{m-1-k}\eta^k\right) \\ & + \left(\sum_{k=0}^{m-2} b_k \zeta^{m-2-k}\eta^k\right)(\tau + b\zeta + c\eta). \end{aligned}$$

By Theorem 3, if the problem (P) is C^∞-well posed then A will be hyperbolic with respect to $(1, \delta, 0)$, for δ small enough and positive. Using the same arguments as in Theorem 2, we will have $A_{m-1} \prec A_m$, i.e., there exists a constant C such that, for all $\xi \in \mathbf{R}^3$,

$$|A_{m-1}(\xi)| \leqslant C|\tilde{A}_m(\xi)|. \tag{2}$$

But, if we consider only the higher exponents of η,

$$\begin{aligned} |\tilde{A}_m(\xi)|^2 &= |A_m(\xi)|^2 + |\partial_\tau A_m(\xi)|^2 + |\partial_\zeta A_m(\xi)|^2 + |\partial_\eta A_m(\xi)|^2 + \cdots \\ &= \left|ca_{m-1}\eta^m + ((\tau + b\zeta)a_{m-1} + c\zeta a_{m-2})\eta^{m-1} + \cdots\right|^2 \\ &\quad + \left|a_{m-1}\eta^{m-1} + \cdots\right|^2 + \left|(ca_{m-2} + ba_{m-1})\eta^{m-1} + \cdots\right|^2 \\ &\quad + \left|mca_{m-1}\eta^{m-1} + \cdots\right|^2 + \cdots \end{aligned}$$

and

$$|A_{m-1}(\xi)| = \left|(da_{m-1} + cb_{m-2})\eta^{m-1} + \cdots\right|.$$

So, if

$$a_{m-1} = a_{m-2} = 0 \quad \text{when } cb_{m-2} \neq 0, \tag{3}$$

the inequality (2) will not hold for η big enough. We can then conclude that the contrary of (3) is a necessary condition for the problem (P) to be C^∞-well posed, thus showing that the condition of Hasegawa is not sufficient.

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