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THE ASYMPTOTIC-NORMING AND THE RADON-NIKODYM PROPERTIES ARE EQUIVALENT IN SEPARABLE BANACH SPACES

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ABSTRACT. We show that the asymptotic-norming and the Radon-Nikodym properties are equivalent, settling a problem of James and Ho [**9**]. In the process, we give a positive solution to two questions of Edgar and Wheeler [**6**] concerning Cech-complete Banach spaces. We also show that a separable Banach space with the Radon-Nikodym property semi-embeds in a separable dual whenever it has a norming space not containing an isomorphic copy of l_1. This gives a partial answer to a problem of Bourgain and Rosenthal [**3**].

Introduction. Let X be a Banach space. We recall that X is said to have *the point of continuity property* (resp. *the Radon-Nikodym property*) if every weakly closed bounded subset of X has a point of weak to norm continuity (resp. a denting point). A separable Banach space X is said to have the *asymptotic norming property* if there exists a separable Banach space Y such that X is (isomorphic to) a subspace of Y^* which verifies the following property:

$$\text{(A.N.P.)} \qquad \begin{array}{l} \text{if } (x_n) \subseteq X, x_n \xrightarrow{w^*} y^* \text{ and } \|x_n\| \to \|y^*\|, \\ \text{then } \lim_n \|x_n - y^*\| = 0. \end{array}$$

In [**9**], James and Ho introduced the asymptotic norming property, proved that it implies the Radon-Nikodym property and asked whether the two properties are equivalent. To prove this conjecture we recall that Davis and Johnson [**5**] showed that for every separable subspace X of Y^*, the latter can be renormed in such a way that the conclusion of (A.N.P.) holds provided y^* is assumed to be in X. The only missing ingredient in the equivalent norm is then the term that forces y^* to be in X. On the other hand, the authors proved in [**7**] the following

THEOREM 1 [**7**]. *Let X be a separable Banach space. Then X has the point of continuity property (resp. the Radon-Nikodym property) if and only if X embeds isometrically in the dual of a separable Banach subspace Y of X^* in such a way that $Y^*\backslash X = \bigcup_n K_n$ where each K_n is weak*-compact (resp. weak*-compact and convex) in Y^*.*

It is easy to see that if the convex K_n's can be chosen to be a strictly positive distance away from X, then the distance of the elements of Y^* to the K_n's (made into a suitable seminorm) would give the missing ingredient that forces y^* to be in X.

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The problem of the existence of such K_n's coincide with a question of Edgar and Wheeler [**6**] on strongly Čech-complete Banach spaces. The following theorem gives a positive answer to these questions.

THEOREM (1) BIS. *Let X be a separable Banach space. Then X has the point of continuity property (resp. the Radon-Nikodym property) if and only if X embeds in the dual of a separable Banach space Y in such a way that $Y^*\backslash X = \bigcup_n K_n$ where each K_n is weak*-compact (resp. weak*-compact and convex) satisfying $d(K_n, X) > 0$.*

The proof will be broken into several lemmas. We shall need the following notations and terminology: If C is a subset of a dual space Y^* we shall denote by $\overline{C}^*$ (resp. $\overline{C}$) its weak*-closure (resp. its norm closure). The distance between two subsets C and D of Y^* will be denoted by $d(C, D) = \inf\{\|x - y\|;\ x \in C,\ y \in D\}$. If Δ is any metric on Y^*, $l \in Y^*$ and $\rho > 0$, then $B_\Delta(l, \rho)$ will be the Δ-open ball $\{y \in Y;\ \Delta(y, l) < \rho\}$. If Δ is induced by the norm we shall simply write $B(l, \rho)$. The closed unit ball of a Banach space Z will be denoted B_Z.

If now X is a subspace of Y^* and L is a w^*-compact subset of Y^* which is disjoint from X, we shall say that L is *ρ-bad with respect to X* for some $\rho > 0$ if

(i) for each $\varepsilon > 0$, the set $L_\varepsilon = \{l \in L;\ d(l, X) \le \varepsilon\}$ is w^*-dense in L;

(ii) the set $L^\rho = \{l \in L;\ d(l, X) \ge \rho\}$ is also w^*-dense in L.

LEMMA (1). *Let X be a subspace of the dual of a separable Banach space Y, such that $\overline{B}_X^* = B_{Y^*}$. Let L be a w^*-compact subset of $\theta \cdot B_{Y^*}$ ($\theta < 1$) which is ρ-bad with respect to X for some $\rho > 0$. Let K be a w^*-compact subset of B_{Y^*} which is disjoint from X. Then for each ε such that $0 < \varepsilon < 1 - \theta$ and each w^*-open subset V of B_{Y^*} with $V \cap L \neq \emptyset$, there exist $l \in L$, $x \in X$ such that*

(i) $\|x\| \le \varepsilon$,

(ii) $l + x \in V$,

(iii) $l + x \notin K$,

(vi) $d(l + x, X) \ge \rho - \varepsilon$.

PROOF. Choose l_1 in L such that $l_1 \in V$ and $d(l_1, X) < \varepsilon/2$. Consider x_1 and then (y_n) in X such that $l_1 = w^*\text{-}\lim_n(x_1 + y_n)$ with $\|y_n\| \le \varepsilon/2$. Note that $\|x_1\| \le \theta + \varepsilon/2$, hence $x_1 + y_n \in B_X$. Choose now n_0 such that $x_1 + y_{n_0} \in V$. We have that $x_1 + y_{n_0} \notin K$. On the other hand, $w^*\text{-}\lim_n(l_1 - y_n + y_{n_0}) = x_1 + y_{n_0}$ and $\|l_1 - y_n + y_{n_0}\| \le 1$, hence, for a large enough n_1, we have $(l_1 - y_{n_1} + y_{n_0}) \in V$ and $(l_1 - y_{n_1} + y_{n_0}) \notin K$.

Since L is ρ-bad, choose $(l_m) \subseteq L$, $d(l_m, X) \ge \rho$ such that $l_1 = w^*\text{-}\lim(l_m)$. That is, for a large enough m_0, we have $(l_{m_0} - y_{n_1} + y_{n_0}) \in V$ and $(l_{m_0} - y_{n_1} + y_{n_0}) \notin K$. Now take $l = l_{m_0}$ and $x = (-y_{n_1} + y_{n_0})$. They clearly verify the claimed properties.

LEMMA (2). *Let X be a separable subspace of the dual of a separable Banach space Y such that B_X is a w^*-G_δ, w^*-dense in B_{Y^*}. Then for each $\theta < 1$, θB_{Y^*} contains no ρ-bad w^*-compact sets with respect to X for any $\rho > 0$.*

PROOF. Write $B_{Y^*}\backslash B_X = \bigcup_n K_n$ where (K_n) is an increasing sequence of w^*-compact sets. Let Δ be a distance defining the w^*-topology on B_{Y^*} and let (z_n) be a dense sequence in B_X. Suppose L is a weak*-compact subset of $\theta \cdot B_{Y^*}$ ($\theta < 1$) which is ρ-bad with respect to X for some $\rho > 0$.

Let l_0 be any point in L and let $V_0 = B_\Delta(l_0, 1)$ and $0 < \varepsilon_0 < \inf(1-\theta, \rho/2)$. Use Lemma (1) to obtain $x_0 \in X$, $\|x_0\| \leq \varepsilon_0$ and $l_0' \in L$ such that $l_0' + x_0 \in V_0$, $l_0' + x_0 \notin K_0$ and $d(l_0' + x_0, X) \geq \rho - \varepsilon_0 > \rho/2$.

Now set $L_1 = L + x_0$, $\theta_1 = \theta + \varepsilon_0 < 1$ and $l_1 = l_0' + x_0$.

Note that L_1 is also ρ-bad with respect to X and $L_1 \subseteq \theta_1 \cdot B_{Y^*}$. Let V_1' be a w^*-open subset of V_0 containing l_1 such that $\overline{V}_1' \cap (K_0 \cup \overline{B}^*(z_0, \rho/2)) = \emptyset$. Set $V_1 = V_1' \cap B_\Delta(l_1, 1/2)$ and $\varepsilon_1 < \inf(1-\theta_1, \rho/2)$ and apply again Lemma 1 to obtain $x_1 \in X$, $\|x_1\| \leq \varepsilon_1$, $l_1' \in L_1$, $l_1' + x_1 \in V_1$, $l_1' + x_1 \notin K_1$ and $d(l_1' + x_1, X) \geq \rho - \varepsilon_1 > \rho/2$.

By induction, we get a decreasing sequence (V_n) of w^*-open subsets of B_{Y^*} and a sequence (l_n) of vectors such that

(i) $l_n \in V_n$ for each n,

(ii) $\operatorname{diam}_\Delta(\overline{V}_n^*) \leq 2^{-n}$,

(iii) $\overline{V}_n^* \cap (K_{n-1} \cup (\bigcup_{j=0}^{n-1} \overline{B}^*(z_j, \rho/2))) = \emptyset$.

It follows that the w^*-limit l_∞ of (l_n) can neither be in B_X nor in any of the K_n's, which is obviously a contradiction since $l_\infty \in B_{Y^*}$.

LEMMA (3). *Let X be a separable subspace of the dual of a separable Banach space Y such that B_X is a w^*-G_δ, w^*-dense in B_{Y^*}. Let L be a subset of Y^* which is disjoint of X. Then:*

(i) *If L is w^*-compact, there exists a w^*-open set V such that $L \cap V \neq \emptyset$ and $d(L \cap \overline{V}^*, X) > 0$.*

(ii) *If L is w^*-compact and convex, there exists a w^*-open half-space V such that $L \cap V \neq \emptyset$ and $d(L \cap \overline{V}^*, X) > 0$.*

PROOF. We first claim that there exists $\varepsilon > 0$ such that the set $L_\varepsilon = \{l \in L;\ d(l, X) \leq \varepsilon\}$ is not w^*-dense in L. Indeed, suppose not. We can assume without loss that $L \subseteq B_{Y^*}/2$. Now note that $L = \bigcup_n L^n$ where $L^n = \{l;\ d(l, X) \geq 1/n\}$ since $L \cap X = \emptyset$. It follows that there exists m such that $\overline{L^m}^*$ has a nonempty interior V_0 in the w^*-topology relative to L. It follows that $\overline{V}_0^*$ is a $1/m$-bad set with respect to X which clearly contradicts Lemma (2).

In case (i) we take V to be a w^*-open subset of Y^* such that $V \cap L = L \backslash \overline{L}_\varepsilon^*$ which is nonempty.

In case (ii), note that $\overline{L}_\varepsilon^*$ is also convex, hence, any w^*-open half-space V that separates any point l in $L \backslash \overline{L}_\varepsilon^*$ from $\overline{L}_\varepsilon^*$ will do the job.

LEMMA (4). *Let X be a separable subspace of the dual of a separable Banach space Y such that B_X is a w^*-G_δ set which is w^*-dense in B_{Y^*}. Let L be a subset of Y^* which is disjoint of X. Then:*

(i) *If L is w^*-compact, there exists a countable collection of w^*-compact sets (L_n) whose union is L such that $d(L_n, X) > 0$ for each n.*

(ii) *If L is w^*-compact and convex, there exists a countable collection of w^*-compact convex sets whose union is L such that $d(L_n, X) > 0$ for each n.*

PROOF. (i) By transfinite induction, we define a decreasing family (K_α) of w^*-compact subsets of L in the following manner:

(a) $K_0 = L$.

(b) If $\alpha = \beta + 1$ and K_β nonempty, apply Lemma (3) to K_β to obtain a w^*-open set V_β such that $K_\beta \cap V_\beta \neq \emptyset$ and $d(K_\beta \cap \overline{V}_\beta^*, X) > 0$. Set $K_\alpha = K_\beta \backslash V_\beta$.

(c) If α is a limit ordinal, set $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$.

Since L is w^*-metrizable there exists $\gamma < \Omega$ (the first uncountable ordinal) such that $K_\gamma = \emptyset$. It is clear that $L = \bigcup_{\alpha<\gamma} K_\alpha \cap \overline{V}^*_\alpha$ and $L_\alpha = K_\alpha \cap \overline{V}^*_\alpha$ is a strictly positive distance away from X for each $\alpha < \gamma$.

(ii) If L is also convex, then V can be taken to be a w^*-open half-space by Lemma (3), hence each $L_\alpha = K_\alpha \cap \overline{V}^*_\alpha$ is then w^*-compact and convex.

The following is now immediate:

THEOREM (1) TER. *Let X be a separable subspace of the dual of a separable Banach space Y such that B_X is w^*-dense in B_{Y^*}. If $Y^*\backslash X = \bigcup_n K_n$ where each K_n is w^*-compact (resp. w^*-compact and convex), then $Y^*\backslash X = \bigcup_n K'_n$ where each K'_n is w^*-compact (resp. w^*-compact and convex) such that $d(K'_n, X) > 0$.*

PROOF OF THEOREM (1) BIS. If X is a separable Banach space with the point of continuity property, apply Theorem (1) to get a separable Banach subspace Y of X^* such that X is a subspace of Y^* verifying $Y^*\backslash X = \bigcup_n K_n$ where each K_n is w^*-compact. It follows that B_X is a w^*-G_δ, w^*-dense subset of B_{Y^*}. Apply now Theorem (1) ter to get the conclusion.

If X has the Radon-Nikodym property, each K_n is then convex, and Theorem (1) ter applies again and gives the claimed result.

The following corollary answers two questions of Edgar and Wheeler [**6**]:

COROLLARY (5). (a) *A separable Banach space X has the point of continuity property and its dual X^* is separable if and only if $X^{**}\backslash X$ is the countable union of w^*-compact sets (K_n) such that $d(K_n, X) > 0$.*

(b) *A separable Banach space X has the Radon-Nikodym property and its dual X^* is separable if and only if $X^{**}\backslash X$ is the countable union of w^*-compact convex sets (K_n) such that $d(K_n, X) > 0$.*

PROOF. In view of the results of [**6** and **7**] the space Y mentioned in Theorem (1) can be taken in this case to be the separable dual X^*.

Theorem (1) bis and the proof of Theorem 4.14 of [**6**] applied to Y instead of X^* gives the following

COROLLARY (6). *A separable Banach space X has the point of continuity property if and only if there exists a separable Banach space Y and a family of norm one vectors $\{y_{n,i};\ 1 \le i \le m_n, n \in \mathbf{N}\}$ in Y such that*

$$X = \left\{ y^* \in Y^*; \varliminf_n \max_{1\le i\le m_n} |y^*(y_{n,i})| = 0 \right\}.$$

The following settles a question of James and Ho [**9**]:

THEOREM (2). *A separable Banach space X has the Radon-Nikodym property if and only if it has the asymptotic-norming property.*

PROOF. Suppose that X has the Radon-Nikodym property. Apply Theorem (1) bis to obtain a separable Banach space Y such that X is a subspace of Y^* verifying $Y^*\backslash X = \bigcup_n K_n$ where each K_n is w^*-compact convex and $d(K_n, X) \ge \varepsilon_n > 0$. Following Davis and Johnson [**5**], let $(E_n)_n$ be an increasing sequence of finite-dimensional subspaces of X such that $X = \overline{\bigcup_n E_n}$ and define the seminorm $|||x||| = \sum_n 2^{-n} d(x, E_n)$. Now let $!\ !_n$ be the seminorm defined by

$$!x!_n = d(x, \mathbf{R}_+ K_n) + d(x, -\mathbf{R}_+ K_n)$$

and set $!x! = \sum_n 2^{-n} !x!_n$. Finally, let $\|x\|_1 = \|x\| + |||x||| + !x!$. Note that $\|x\| \leq \|x\|_1 \leq 7\|x\|$ for each x in Y^* and that $\|\ \|_1$ is w^*-lower semicontinuous, hence, it is a dual norm on Y^*.

Suppose now that $(x_n) \subseteq X$, $y^* \in Y^*$ such that $\|x_n\|_1 \to \|y^*\|_1$ and w^*-$\lim_n(x_n) = y^*$. Since each piece of the norm is w^*-lower semicontinuous we get that $\|x_n\| \to \|y^*\|$, $|||x_n||| \to |||y^*|||$ and $d(x_n, \mathbf{R}_+ K_m) \to d(y^*, \mathbf{R}_+ K_m)$ for each m.

We claim that $y^* \in X$. Indeed, if not, then there exists an m such that $y^* \in K_m$ and $\lim_n d(x_n, \mathbf{R}_+ K_m) = d(y^*, \mathbf{R}_+ K_m) = 0$.

We can then suppose that $\|x_n - \lambda_n k_n\| \leq 1/n$ for some $\lambda_n \in \mathbf{R}_+$ and $k_n \in K_m$. This gives

$$1/n \geq \|x_n - \lambda_n k_n\| = \lambda_n \|x_n/\lambda_n - k_n\| \geq \lambda_n \varepsilon_m.$$

It follows that $\lambda_n \to 0$ and $\|x_n\| \to 0$, a contradiction. Since y^* is now in X, the Davis-Johnson norm insures that $\lim_n \|x_n - y^*\| = 0$.

The converse was proved by James and Ho [**9**]. We sketch an easier proof based on martingales and already used by Davis et al. [**4**]. Let D be a countable dense set in the unit ball of Y. Let (ϕ_n) be an X-valued bounded martingale. Let ϕ_∞ be a w^*-limit of (ϕ_n) which is valued in Y^*. For each $y \in D$, the real-valued martingale $y(\phi_n)$ converges to $y(\phi_\infty)$ outside a set Ω_y of measure zero.

By a lemma of Neveu [**11**], the martingale $\|\phi_n\| = \sup_{y \in D} |y(\phi_n)|$ converges to $\sup_{y \in D} |y(\phi_\infty)| = \|\phi_\infty\|$ outside a set Ω_0 of measure zero. Since X has the asymptotic norming property with respect to Y, we get that $\lim_{n \to \infty} \|\phi_n - \phi_\infty\| = 0$ outside the set $\Omega_0 \cup \bigcup_{y \in D} \Omega_y$ which is of measure zero.

Recall that a bounded linear operator T from a Banach space X into a space Y is said to be a semi-embedding if it is one-to-one and if the image of the unit ball of X by T is norm closed in Y. In [**3**], Bourgain and Rosenthal showed that the $\mathcal{L}_\infty$-spaces constructed by Bourgain and Delbaen [**2**] do not semi-embed in separable duals even though they enjoy the Radon-Nikodym property. On the other hand, they show that the Radon-Nikodym spaces constructed by Johnson and Lindenstrauss [**10**] do semi-embed in separable duals even though they do not embed in such spaces. The following theorem gives a sufficient condition that guarantees such semi-embeddings for Radon-Nikodym spaces. It gives a partial solution to a question of Bourgain and Rosenthal [**3**].

THEOREM (3). *If X is a separable Banach space with a norming space not containing an isomorphic copy of l_1, then X has the Radon-Nikodym property if and only if it semi-embeds in a separable dual.*

First we need the following

LEMMA (5). *Let Y be a separable Banach space not containing an isomorphic copy of l_1. If X is a separable subspace of Y^* with the Radon-Nikodym property such that B_X is w^*-dense in B_{Y^*}, then the orthogonal of X in Y^{**} is w^*-separable.*

PROOF. By a theorem of Bourgain [**1**], B_X is then w^*-dentable in Y^*; that is, every norm closed convex subset of B_X contains w^*-open slices with arbitrarily small diameters. Now, we proceed as in Lemma III.1 of [**7**]: Fix $\varepsilon > 0$ and define inductively a decreasing family of norm-closed convex subsets (F_α) of B_X in the following way:

(i) $F_0 = B_X$.

(ii) If $\alpha = \beta + 1$ and $F_\beta \neq \emptyset$, use the w^*-dentability to find a w^*-open slice S_β of F_β such that $\operatorname{diam}(S_\beta) < \varepsilon$. Set $F_\alpha = F_\beta \backslash S_\beta$.

(iii) If α is a limit ordinal, let $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$.

Since B_X is separable, there exists $\gamma < \Omega$ (the first uncountable ordinal) such that $F_\gamma = \emptyset$ and $F_\beta \neq \emptyset$ for $\beta < \gamma$. Let K_α be the w^*-closure of F_α in Y^* and let H_α be the w^*-open half-space such that $S_\alpha = H_\alpha \cap F_\alpha$. It is clear that

$$B_X \subseteq \bigcap_{\alpha \leq \gamma} \left(K_\alpha \cup \bigcup_{\beta < \alpha} H_\beta \right).$$

Moreover, if x belongs to the set on the right-hand side, then $x \in K_\beta \cap H_\beta$ for some $\beta < \gamma$ which implies that $d(x, B_X) \leq \varepsilon$. It follows that if we repeat the construction for each $\varepsilon = 1/n$ we then get

$$B_X = \bigcap_n \bigcap_{\alpha \leq \gamma_n} \left(K_{\alpha,n} \cup \bigcup_{\beta<\alpha} H_{\beta,n} \right).$$

Since Y is separable, write that $K_{\alpha,n} = \bigcap_m L_{\alpha,n,m}$ where each $L_{\alpha,n,m}$ is a w^*-open half-space in Y^*. It follows that $Y^* \backslash B_X$ and hence $Y^* \backslash X$ is a countable union of w^*-compact convex subsets (K_n) of Y^*. By Theorem (1) ter, we can suppose that $d(K_n, X) > \varepsilon_n > 0$. If π is now the quotient map from Y^* onto Y^*/X, we obtain that $0 \notin \overline{\pi(K_n)}$ for each n, hence, there exists f_n in $(Y^*/X)^* = X^\perp$ such that $f_n > \varepsilon_n$ on $\overline{\pi(K_n)}$. It is now clear that $X = \{y^* \in Y^*;\ f_n(y^*) = 0;\ \forall n \in \mathbf{N}\}$ and that $X^\perp$ is w^*-separable.

PROOF OF THEOREM (3). Since l_1 does not embed in Y, use Odell and Rosenthal's theorem [**12**] to find for each n a sequence $(g_{n,m})$ in Y that converges pointwise on Y^* to f_n. The space X can now be written as $\{y^* \in Y^*;\ \lim_{m\to\infty} g_{n,m}(y^*) = 0;\ \forall n \in \mathbf{N}\}$. We may suppose that $\|g_{n,m}\| \leq 1$ for each n, $m \in \mathbf{N}$. Define now for each $n \geq 1$, the operator $T_n \colon l_1 \to Y$ by $T_n(\alpha_m)_m = \sum_m \alpha_m g_{n,m}$ and let $T_0 \colon l_2 \to Y$ be a dense range operator. Let $T \colon l_2 \oplus (\sum_n \bigoplus l_1)_{l_2} \to Y$ be the unique linear operator whose restrictions to the factor spaces are those given by the sequence $\{T_0, (2^{-n}T_n)_{n\geq 1}\}$. Let $T^* \colon Y^* \to l_2 \oplus (\sum_n \bigoplus l_\infty)_{l_2}$ be the dual operator which is one-to-one since T_0^* is. We claim that T^* is valued in $l_2 \oplus (\sum_n \bigoplus c)_{l_2}$ and that $T^*(B_X)$ is norm-closed. Indeed, we have for each $n \geq 1$ and each $y^* \in Y^*$, $(T_n^*(y^*)) = (g_{n,m}(y^*))_m$ which is convergent to $f_n(y^*)$. Moreover, if $x_l \in B_X$ and $\lim_l T^* x_l = z$, then $z = T^* y^*$ where y^* is a w^*-limit of (x_l) in Y^* since T^* is one-to-one and w^*-to-w^* continuous. Moreover, since for each n, $g_{n,m}(x_l) \to g_{n,m}(y^*)$ uniformly in m, we get that $f_n(x_l) \to f_n(y^*)$ and $f_n(y^*) = 0$ for each n. It follows that $y^* \in B_X$.

The operator T has a separable adjoint, hence the Stegall factorization theorem [**13**] applies and we get a separable Banach space Z with a separable dual such that $T = U \circ V$ where $U \colon Z \to Y$ and $V \colon l_2 \oplus (\sum_n \bigoplus l_1)_{l_2} \to Z$. Note now that $U^*(B_X)$ is norm-closed in Z^* since $T^*(B_X)$ is closed in $l_2 \oplus (\sum_n \bigoplus l_\infty)_{l_2}$. Hence U^* is a semi-embedding of X into the separable dual Z^*.

REMARK. The proof of Theorem (1) bis relies heavily on the fact that the ball of X is a w^*-G_δ in Y^*. Actually the local statement is not true. In a forthcoming paper, we construct a w^*-G_δ subset C of a dual space whose complement is not decomposable into a countable union of w^*-compact sets which are a strictly positive distance away from C. This question is closely related to the problem of minimizing a certain class of functions on the set C. We shall deal with these questions in [**8**].

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