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MINIMUM EIGENVALUES FOR POSITIVE, ROCKLAND OPERATORS[1]

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ABSTRACT. Let L be a positive, Rockland operator of homogeneous degree γ. The minimum eigenvalue of $d\pi(L)$ increases as the γth power of the homogeneous distance from the origin of the orbit corresponding to π.

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let A be a diagonalizable operator on $\mathfrak{g}$ with eigenvalues $1 = \gamma_1 < \cdots < \gamma_p$, such that for each $r > 0$, the operator $\delta_r = r^A$ is an automorphism of $\mathfrak{g}$ and, hence, determines an automorphism, again denoted by δ_r, of G. Let $|\cdot|$ denote a homogeneous gauge on G; i.e., $|\cdot|$ is a continuous, nonnegative function on G with $|g| = 0$ if, and only if, $g = e$, and satisfying $|\delta_r g| = r|g|$ for $r > 0$. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, and let δ_r^* denote the adjoint of δ_r acting on $\mathfrak{g}^*$. We define $|\cdot|$ on $\mathfrak{g}$ by $|X| = |\exp X|$ and on $\mathfrak{g}^*$ as follows: fix a basis $\{X_1, \ldots, X_d\}$ of g consisting of eigenvectors of A, and let $\{X_1^*, \ldots, X_d^*\}$ be the dual basis of $\mathfrak{g}^*$. For $\xi \in \mathfrak{g}^*$ we set $|\xi| = |\Sigma(\xi, X_i)X_i|$. One easily sees that $|\delta_r X| = r|X|$ and $|\delta_r^* \xi| = r|\xi|$ for $X \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$, and $r > 0$.

Let $g \to \mathrm{Ad}^*g$ be the coadjoint representation of G on $\mathfrak{g}^*$, and for $\xi \in \mathfrak{g}^*$ let $O(\xi) = \mathrm{Ad}^*G \cdot \xi$. Set

$$|O(\xi)| = \inf\{|\xi'| \;\; |\xi' \in O(\xi)\}.$$

By Kirillov theory [1] the equivalence classes of irreducible unitary representations of G, $\hat{G}$, can be identified with $\{O(\xi) | \xi \in \mathfrak{g}^*\}$. Given $\xi \in \mathfrak{g}^*$, we denote by π_ξ the element of $\hat{G}$ corresponding to $O(\xi)$, and by $d\pi_\xi$ the resulting representation of $\mathscr{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

An element L of $\mathscr{U}(\mathfrak{g})$ is said to be homogeneous of degree γ if $L(f \circ \delta_r) = r^\gamma (Lf) \circ \delta_r$ for all smooth functions of f. As in [2], a homogeneous operator L is called a Rockland operator if $d\pi_\xi(L)$ is injective (on the space of smooth vectors) for each $\xi \neq 0$. L is said to be positive if $(Lf, f) \geqslant 0$ for all $f \in C_c^\infty(G)$. The proof by Nelson and Steinspring [6] that $d\pi(L)$ is essentially selfadjoint for any elliptic operator and any unitary representation π uses, in fact, only the hypoellipticity of L. Thus, by the theorem of Helffer and Nourrigat [3] $d\pi_\xi(L)$ is essentially selfadjoint for any Rockland operator. Therefore, if L is a positive, Rockland operator, both L and $d\pi_\xi(L)$ are infinitesimal generators of contraction semigroups. In [2] Folland

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and Stein show that the densities of the semigroup generated by L, $\{P_t\}_{t>0}$, are Schwartz functions on G. One can easily show that the infinitesimal generator of $\{\pi_\xi(P_t)\}_{t>0}$ is $d\pi_\xi(L)$. Since π_ξ, integrated to $L^1(G)$, maps into compact operators, $d\pi_\xi(L)$ has as eigenvalues a discrete subset $\sigma_\xi(L) \subset (0, \infty)$ for each $0 \neq \xi \in \mathfrak{g}^*$.

THEOREM. *Let L be a positive, Rockland operator of degree γ. There is a $c > 0$ such that, for all $\xi \in \mathfrak{g}^*$,* $\min\{\alpha \in \sigma_\xi(L)\} \geqslant c|O(\xi)|\mathfrak{g}^\gamma$.

PROOF. Define δ_r^* on $\mathfrak{g}^*$ by $(\delta_r^*\xi, X) = (\xi, \delta_r X)$. Then

(i) $$d\pi_{\sigma_r^*\xi}(L) \cong d_{\pi_\xi}(L \circ \delta_r) = r^\gamma d\pi_\xi(L).$$

Indeed, if $\alpha \in \operatorname{Aut}(G)$, we may also regard α as an automorphism of $\mathfrak{g}$ and similarly define $\alpha^* \in \operatorname{End}(\mathfrak{g}^*)$. If $\pi \in \hat{G}$ we set $\pi^\alpha(x) = \pi(\alpha(x))$. One can easily verify from Kirillov theory that $\pi_\xi^\alpha = \pi_{\alpha^*\xi}$. This implies (i) for $\alpha = \delta_r$ and L homogeneous of degree γ.

From (i) one easily shows

(ii) $$\min\left\{\alpha \in \sigma_{\delta_s^*\xi}(L)\right\} \geq s^\gamma \min\left\{\alpha \in \sigma_\xi(L)\right\}$$

by merely noting that f is an eigenvector $d\pi_{\delta_s^*\xi}(L)$ if, and only if, $f \circ \delta_{1/s}$ is an eigenvector for $d\pi_\xi(\delta_s L) = s^\gamma d\pi_\xi(L)$, and the corresponding eigenvalues differ by factor of s^γ.

We now set $B = \{\xi \in \mathfrak{g}^* | 1 = |\xi| = |O(\xi)|\}$. It is clear that B is compact and for each $\xi \in \mathfrak{g}^*$ there is a $\xi' \in B$ such that $O(\xi) = O(\delta_s^*\xi')$, where $s = |O(\xi)|$. Also, one has

(iii) $$\inf\left\{\alpha | \alpha \in \sigma_\xi(L), \xi \in B\right\} > 0.$$

To see this, use induction on k, where $B_k = \{\xi \in B | \dim O(\xi) = 2k\}$. The case $k = 0$ is obvious, since for $\xi \in B_0$, π_ξ is a character on G and, thus, $\sigma_\xi(L) = \{d\pi_\xi(L)\}$.

Assume $\inf\{\alpha | \alpha \in \sigma_\xi(L), \xi \in B_l, l < k\} \geqslant 0$ and suppose $\alpha_n \in \sigma_{\xi_n}(L)$, $\xi_n \in B_k$, $\alpha_n \to 0$, and $\xi_n \to \xi_0$ in $\mathfrak{g}^*$. As in [**1**, p. 105] pick bases $\{X_1^{(n)}, \ldots, X_d^{(n)}\}$ of $\mathfrak{g}$ so that $\{X_1^{(n)}, \ldots, X_{d-k}^{(n)}\}$ is a basis for a polarization $\mathfrak{h}_n$ of ξ_n, so that $X_i = \operatorname{Lim} X_i^{(n)}$ exists for $1 \leqslant i \leqslant d$ and $\{X_1, \ldots, X_{d-k}\}$ is contained in a basis for a polarization of ξ_0. Define β_n: $\mathbf{R}^{d-k} \times \mathbf{R}^k \to G$ by

$$\beta_n(\mathbf{t}, \mathbf{s}) = \exp t_1 X_1^{(n)} \cdots \exp s_k X_d^{(n)},$$

where $\mathbf{t} = (t_1, \ldots, t_{d-k})$ and $\mathbf{s} = (s_1, \ldots, s_k)$. Define $\tilde{\beta}_n$: $L^2(H_n \setminus G, \chi_{\xi_n}) \to L^2(\mathbf{R}^k)$ by

$$(\tilde{\beta}_n f)(\mathbf{s}) = f(\beta_n(0, \mathbf{s})).$$

With the correct normalization of Lebesgue measure on $\mathbf{R}^k$, $\tilde{\beta}_n$ is an isometric isomorphism. Thus $\pi_{\xi_n} \simeq \tilde{\pi}_{\xi_n} = \tilde{\beta}_n \circ \pi_\xi \circ \tilde{\beta}_n^{-1}$. Also, one has that $\tilde{\pi}(g) = \operatorname{Lim} \tilde{\pi}_{\xi_n}(g)$ exists for each $g \in G$.

Let $\{P_t\}_{t>0}$ be the semigroup generated by L, and set

$$R_1 = \int_0^\infty e^{-t} P_t \, dt.$$

Then $R_1 = (I + L)^{-1}$ (cf. [7]), and the assumptions imply that $\beta_n = 1/(1 + \alpha_n) \in \sigma_{\xi_n}(R_1)$, the spectrum of $\pi_{\xi_n}(R_1)$. Thus, $\sup\{\beta \in \sigma_{\xi_n}(R_1)\} \geqslant 1$. We will show that this is impossible.

In [5, Proposition 6.3], it is shown that, by passing to a subsequence if necessary, there are only finitely many orbits in $\mathfrak{g}^*$, with dimension $2k$, that are limit points in $\hat{G}$ of $\{O(\xi_n)\}$. Denote this set by $\mathscr{A}_1$. Let $\mathscr{A}_2$ be the lower-dimensional orbits that are limit points of $\{o(\xi_n)\}$, and let $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$. Then, by [4],

$$\lim_n \left\|\pi_{\xi_n}(R_1)\right\| = \sup_{o(\xi) \in \mathscr{A}} \left\|\pi_\xi(R_1)\right\|.$$

Thus,

$$\limsup_n \left\{\beta \in \sigma_{\xi_n}(R_1)\right\} = \max\left\{\sup_{\mathscr{A}_1} \left\|\pi_\xi(R_1)\right\|, \sup_{\mathscr{A}_2} \left\|\pi_\xi(R_1)\right\|\right\} < 1,$$

since $\mathscr{A}_1$ is finite and the orbits in $\mathscr{A}_2$ have dimension less than $2k$.

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