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THE SUM OF TWO RADON-NIKODYM-SETS NEED NOT BE A RADON-NIKODYM-SET

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ABSTRACT. It was shown by C. Stegall that, if C is a Radon-Nikodym-set and K weakly compact, then $K + C$ is a Radon-Nikodym-set. We show that there are closed, bounded, convex Radon-Nikodym-sets C_1 and C_2 such that $C_1 + C_2$ is closed but contains an isometric copy of the unit-ball of c_0. In fact, we give two examples, one following the lines of one due to McCartney and O'Brian, the other due to Bourgain and Delbaen. We also give an easy example of a non-Radon-Nikodym-set C such that, for every $\varepsilon > 0$, there is a Radon-Nikodym-set C_ε such that C is contained in the sum of C_ε and the ball of radius ε.

1. Introduction.

1.1. DEFINITION [9]. A closed, bounded, convex subset C of a Banach space X is called a Radon-Nikodym-set (abbreviated RN-set) if, for every probability space (Ω, Σ, P) and every X-valued vector measure $m: \Sigma \to X$ such that the average of m is in C (i.e. $P(E)^{-1}m(E)$ is in C for every $E \in \Sigma$), there exists a Bochner integrable function $f: \Omega \to X$ such that

$$m(E) = \int_E f\,dP$$

for all $E \in \Sigma$.

In the present paper we supply the "local" point of view, i.e. we consider RN-sets instead of spaces with RNP (X has RNP if the unit ball of X is an RN-set). The break through for this local point of view was J. Bourgain's "internal" proof of the Lindenstauss-Trojanski theorem [1]. In fact, he proved much more:

1.2. THEOREM (BOURGAIN, PHELPS; SEE [3, THEOREM 3.5.4]). *A closed, bounded, convex subset C of a Banach space X is an RN-set iff for every closed, bounded, convex subset D of C the functions in* X^* *which strongly expose D are a dense* G_δ.

1.3. DEFINITION. If A is a subset of a Banach space X and $f \in X^*$, then f *strongly exposes* A if, for every sequence $(x_n)_{n=1}^\infty$ in A such that

$$\lim_n f(x_n) = \sup\{f(x): x \in A\},$$

it follows that $(x_n)_{n=1}^\infty$ is Cauchy. If A is closed there exists a unique *strongly exposed* point $x_0 \in A$ such that

$$f(x_0) = \sup\{f(x): x \in A\}.$$

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1.4. Note that f strongly exposes A iff f determines "slices of A of arbitrarily small diameter" (for definitions and notations we refer to [**3**, Chapter 3]). Having this in mind one easily verifies that for bounded A the set of strongly exposing functionals is automatically a G_δ.

1.5. Let us now consider the question of whether the sum of two RN-sets is an RN-set. The following result was proved by C. Stegall ([**9**, p. 46]; in fact C. Stegall states there that this result was also known by J. Bourgain);

1.6. PROPOSITION [**9**, p. 46]. *Let $C \subseteq X$ be a (closed, convex, bounded) RN-set and $W \subseteq X$ be weakly compact and convex. Then $D = C + W$ is an RN-set.* $\square$

We shall prove here a slightly more general version. We thank J. J. Uhl who pointed out to us how to apply a "Lindenstrauss compactness argument" used in [**10**].

PROPOSITION 1.6(a). *Let $(X, \|\cdot\|)$ be a Banach space and let τ be a Hausdorff vector space topology on X coarser than the norm topology. Let $C \subseteq X$ and $W \subseteq X$ be (convex, bounded) RN-sets such that C is τ-closed and W is τ-compact. Then $W + C$ is an RN-set.*

PROOF. First note that $W + C$ is τ-closed and therefore normclosed. Let T: $L^1(0,1) \to X$ be such that $T(\chi_A/\lambda(A)) \in W + C$ for every measurable A, $\lambda(A) > 0$. For every $n \geq 0$ let A_n be the algebra generated by the nth dyadic partition of $[0,1]$, $\{I_n^1, \ldots, I_n^{2^n}\}$. For every $1 \leq j \leq 2^n$ we have that

$$T(2^n \chi_{I_n^j}) = U_n(2^n \chi_{I_n^j}) + V_n(2^n \chi_{I_n^j})$$

(pick any choice such that the first member is in W and the second in C). This enables us to define two linear operators U_n and V_n: $L^1(A_n) \to X$ such that $T = U_n + V_n$ on $L^1(A_n)$ and U_n and V_n have their average ranges in W and C, respectively.

Let $\tilde{U}_n$: $L^1(0,1) \to X$ be U_n composed with the canonical projection (i.e. the conditional expectation) from $L^1(0,1)$ onto $L^1(A_n)$. Then it follows from a well-known argument (due to Lindenstrauss) that the τ-compactness of W allows us to find a clusterpoint U of the sequence $(\tilde{U}_n)_{n=1}^\infty$ (with respect to the strong topology in the space of operators from L^1 to (X, τ)). Clearly, U has its average range in W. Defining $V = T - U$ we conclude from the τ-closedness of C that V has its average range in C. By hypothesis U and V have representing functions f and g, resp., in $L^\infty([0,1]; X)$, hence $T = U + V$ has $f + g$ as representing function. This shows that $C + W$ is an RN-set. $\square$

The question of whether one may weaken the assumption in 1.6 to require only that W be an RN-set remained open and we shall show that this is in fact not possible even in the case where the sum is closed. We begin with a positive result.

1.7. PROPOSITION. *Let C_1, C_2 be closed, bounded, nonempty convex subsets of a Banach space X and let C be the closure of the sum $C_1 + C_2$, i.e. $C = \overline{C_1 + C_2}$, where*

$C_1 + C_2 = \{x_1 + x_2\colon x_1 \in C_1,\ x_2 \in C_2\}$. *If* $\mathscr{F}$, $\mathscr{F}_1$, $\mathscr{F}_2$ *denote the sets of strongly exposing functionals of* C, C_1, C_2, *resp., then*

$$\mathscr{F} = \mathscr{F}_1 \cap \mathscr{F}_2.$$

In particular, if C_1, C_2 *are RN-sets, then* $\mathscr{F}$ *is a dense* G_δ*-subset of* X^*.

PROOF. Let $\varepsilon > 0$. If $f \in \mathscr{F}$, then f determines a slice of C of diameter less than ε. This clearly implies that f determines slices of C_1 and C_2 of diameter less than ε. Conversely, if f determines slices of C_1 and C_2 of diameter less than ε, then f determines a slice of C of diameter less then or equal to 2ε. $\square$

In particular, if C_1, C_2 are RN-sets we see that $C = \overline{C_1 + C_2}$ has many strongly exposing functionals and is therefore (by well-known arguments) the closed convex hull of its strongly exposed points. The crux is that we are not able to infer this for all closed, convex, bounded subsets of C.

Let us point out here a connection with **[8]**, where a $(1 + \varepsilon)$-equivalent renorming $|||\cdot|||$ of c_0 was constructed, such that the unit-ball with respect to $|||\cdot|||$ is the closed, convex hull of a countable family of strongly exposed points (the strong exposedness there is even uniform and "linear"). By modifying the construction slightly one may also ensure that the functionals which strongly expose the unit-ball with respect to $|||\cdot|||$ are dense in l^1. This shows that there are sets which "look nice at their boundary" but contain subsets which lack this property (e.g. the unit-ball of c_0 with respect to the original norm in the present example). Of course, the subsequent counterexample will also provide an example of this, in view of Proposition 1.7.

2. The example. It is based on the construction of MacCartney and O'Brian **[7]** and we shall use essentially the version given by Johnson and Lindenstrauss **[6]**. First we need a lemma which is easy and well known but which we prove for the sake of completeness. Let F be the positive face of the unit-ball of l^1, i.e.

$$F = \left\{(\lambda_n)_{n=1}^{\infty} \in l^1\colon \Sigma\lambda_n = 1 \text{ and } \lambda_n \geqslant 0 \text{ for all } n\right\}.$$

LEMMA 2.1. *There is a quotient map* $p\colon l^1 \to c_0$ *which maps* F *onto the closed unit-ball of* c_0.

PROOF. Let $(s_n)_{n=1}^{\infty}$ be an enumeration of all vectors of the form $(\varepsilon_1, \ldots, \varepsilon_k, 0, \ldots)$ where $k \in \mathbf{N}$ and $\varepsilon_i = \pm 1$. Let p take the nth unit-vector e_n of l^1 to s_n. Clearly p is a quotient map from l^1 onto c_0 (i.e. takes the open unit-ball onto the open unit-ball). To show that p in fact takes F onto the closed unit-ball, observe that if $x \in c_0$, $\|x\| \leqslant 1$, then we may write

$$x = \sum_{i=1}^{\infty} \lambda_i z_i,$$

where $0 \leqslant \lambda_i \leqslant 1$, $\Sigma\lambda_i = 1$, and z_i are finite vectors with $\|z_i\| \leqslant 1$. Therefore, every z_i may be written as a convex combination of the s_n, i.e. for every i we may find $0 \leqslant \lambda_1^i, \ldots, \lambda_{n(i)}^i \leqslant 1$ with

$$\sum_{n=1}^{n(i)} \lambda_n^i = 1$$

and

$$z_i = \sum_{n=1}^{n(i)} \lambda_n^i s_n = p\left(\sum_{n=1}^{n(i)} \lambda_n^i e_n\right).$$

Hence,

$$x = p\left(\sum_{i=1}^{\infty} \sum_{n=1}^{n(i)} \lambda_i \cdot \lambda_n^i \cdot e_n\right)$$

where the element on the right-hand side is a member of F. □

2.2. From now on, let $p\colon l^1 \to c_0$ be a fixed quotient map which takes F onto the closed unit-ball of c_0. Let X be the space $l^1 \oplus c_0$ and put

$$C^1 = \{(\xi, \eta) \in X\colon \xi \in F \text{ and } p(\xi) = \eta\}$$

and

$$C^2 = (-\xi, \eta) \in X\colon \xi \in F \text{ and } p(\xi) = \eta\}.$$

Hence, C^1 is the subset of the Graph of p lying over F while C^2 is the subset of the Graph of $-p$ lying over $-F$. Clearly, C^1 and C^2 are RN-sets as the Graph of p is isometric to l^1.

However, as the reader will probably have observed—passing from C^1 and C^2 to the sum $C = C^1 + C^2$—the first coordinate may cancel out and therefore one immediately sees that C contains the ball of radius 2 of c_0 (which we identify with the subspace $\{0\} \times c_0$ of X). In the following proposition we give an explicit description of C.

2.3. Proposition. *$C = C^1 + C^2$ is given by the set of $(\xi, \eta) \in X$ satisfying*

(i) $\sum \xi_i = 0$,

(ii) $\|\xi\| \leq 2$,

(iii) $\|\eta - p(|\xi|)\| \leq 2 - \|\xi\|$,

where $|\xi|$ denotes the element $(|\xi_i|)_{i=1}^{\infty}$ of l^1. Hence, C is a closed, bounded, convex subset of X which contains the subset $D = \{(0, \eta)\colon \|\eta\| \leq 2\}$ and, in particular, C is not an RN-set.

Proof. If (ξ, η) is an element of $C^1 + C^2$, there are ξ_1, ξ_2 in F such that

$$(\xi, \eta) = (\xi^1 - \xi^2, p(\xi^1) + p(\xi^2)).$$

Clearly, (ξ, η) satisfies (i) and (ii). For (iii) let

$$\xi^1 \wedge \xi^2 = (\xi_i^1 \wedge \xi_i^2)_{i=1}^{\infty}$$

and note that

$$\begin{aligned} p(\xi^1) + p(\xi^2) &= p(\xi^1 - \xi^1 \wedge \xi^2) + p(\xi^2 - \xi^1 \wedge \xi^2) + 2p(\xi^1 \wedge \xi^2) \\ &= p(|\xi|) + 2p(\xi^1 \wedge \xi^2). \end{aligned}$$

Noting that $\|\xi^1 \wedge \xi^2\| = 1 - \||\xi|\|/2 = 1 - \|\xi\|/2$ we may conclude that

$$\|\eta - p(|\xi|)\| = 2\|p(\xi^1 \wedge \xi^2)\| \leq 2 - \|\xi\|,$$

hence (ξ, η) also satisfies (iii).

Conversely, if (ξ, η) satisfies (i), (ii) and (iii), find $\zeta \in F$ such that

$$(2 - \|\xi\|)p(\zeta) = \eta - p(|\xi|)$$

which is possible by Lemma 2.1. Define

$$\xi^1 = \xi^+ + (1 - \|\xi\|/2) \cdot \zeta,$$
$$\xi^2 = \xi^- + (1 - \|\xi\|/2) \cdot \zeta,$$

where $\xi^+ = (\xi_i \vee 0)_{i=1}^\infty$ and $\xi^- = (-\xi_i \vee 0)_{i=1}^\infty$. The points $(\xi^1, p(\xi^1))$ and $(-\xi^2, p(\xi^2))$ are elements of C^1 and C^2, respectively, and their sum is (ξ, η) since $\xi^1 - \xi^2 = \xi^+ - \xi^- = \xi$ and

$$\begin{aligned} p(\xi^1) + p(\xi^2) &= p(|\xi|) + (2 - \|\xi\|)p(\zeta) \\ &= p(|\xi|) + \eta - p(|\xi|) = \eta. \quad \square \end{aligned}$$

3.4. REMARK. In this context one may also consider the following question (and in fact the author did so): Suppose a closed, bounded, convex subset C of a Banach space X is such that, for every $\varepsilon > 0$, we may find an RN-set C_ε such that

$$(*) \qquad C \subset C_\varepsilon + \varepsilon \cdot B(X)$$

where $B(X)$ is the unit-ball of X. Does this imply that C is a RN-set?

The answer is no and the counterexample embarassingly simple: Let X again be $l^1 \oplus c_0$ and $C = \{(0, \eta): \|\eta\| \leqslant 1\}$. For $\varepsilon > 0$ one may choose

$$C_\varepsilon = \{(\xi, \eta): \|\xi\| \leqslant \varepsilon \text{ and } p(\varepsilon^{-1}\xi) = \eta\}.$$

This example is in fact not very surprising, as Ghoussoub and Johnson have shown [**5**] that there is a RN-operator which does not factor through an RN-space (in fact, they attribute the proof of this part of their paper to M. Talagrand). It is well known that a stability property of the type $(*)$ above is needed (among other things) to be able to apply the Davis-Figiel-Johnson-Pelczynski-factorisation, which apparently does not work for RN-operators.

2.5. To end this section we still observe that in example 2.2 the convex hull of C^1 and C^2 also contains the unit-ball of c_0 and therefore fails to be a RN-set.

4. Another example. There are essentially 2 examples known of RN-spaces which do not embed into a separable dual: the McCartney-O'Brian-construction and the Bourgain-Delbaen-construction. We cannot resist sketching a method of using the Bourgain-Delbaen-construction to give another example of 2 RN-sets C^+ and C^- such that their sum contains the unit-ball of c_0. In the sequel we assume that the reader is familiar with the Bourgain-Delbaen-construction and has a copy on his desk [**2**, p. 26] and we use their notation without further explication.

Let us first give the idea: Recall that Bourgain and Delbaen injected the space $E_n = l^\infty(d_n)$ into l^∞ by leaving the d_n coordinates unchanged and adding a "tail" of coordinates which is skillfully chosen to generate the desired pathologies.

The present idea is to carry out the same construction with "signs changed", i.e. the tail of coordinates added is just the original tail, multiplied by -1. Then we want to take C^+ to be the unit-ball of the original Bourgain-Delbaen-space and C^- to be the unit-ball of this modified construction. Passing to $C^+ + C^-$ we may find pairs such that "tails" cancel out and we end up with the finite vectors, i.e. we find the unit-ball of c_0 in $C^1 + C^2$.

Unfortunately, some difficulties arise in carrying out this idea when one pastes together the injections. Hence we must modify the original construction, inserting some additional "neutral" coordinates.

4.1. We shall now construct two Banach spaces, X^+ and X^-, where X^+ is similar to the Bourgain-Delbaen-space X while X^- is a "changed-signs-version" of X^+. We shall duplicate the construction of X with analogous notation and superscripts $+$ and $-$, respectively. We change the notation only to the extent that the role of d_n in **[2]** is played by $\dim_n$ in our construction. This is because our $\dim_n$ will grow slightly faster than the original d_n.

We start in exactly the same way as **[2]**: Let $\dim_1 = \dim_2 = 1$ and let $i_{1,2}^+$ and $i_{1,2}^-$ be simply the identity on $E_1 = E_2$ which is the one-dimensional space $\mathbf{R}$. We now present the induction step for $n = 2$ to make the idea clear: As in **[2]** we set

$$g_2^+(x) = f_{1,1,1,1,1}^+(x) = ax_1,$$

etc., until

$$g_5^+(x) = f_{1,1,1,-1,-1}^+(x) = -ax_1.$$

For the construction of X^- we simply change signs, i.e.

$$g_2^-(x) = -g_2^+(x) = -ax_1,$$

etc., until

$$g_5^-(x) = -g_5^+(x) = ax_1.$$

In addition, we introduce a new coordinate—number 6 in this case—and let it simply be zero:

$$g_6^+(x) = g_6^-(x) = 0.$$

The dimension of E_3^+ and E_3^- is therefore $\dim_3 = 6$ and we get the injections

$$i_{2,3}^+(x) = (x_1, ax_1, ax_1, -ax_1, -ax_1, 0),$$
$$i_{2,3}^-(x) = (x_1, -ax_1, -ax_1, ax_1, ax_1, 0).$$

This completes the induction step for $n = 2$.

We shall now give the general induction step. Suppose $\dim_m$ $(m \leqslant n)$ is known and the $i_{l,m}^+$ and $i_{l,m}^-$ $(l < m \leqslant n)$ is constructed such that they verify (α) and (β). For $m < n$, $1 \leqslant i \leqslant \dim_m$, $1 \leqslant j \leqslant \dim_n$, $\varepsilon' = \pm 1$, $\varepsilon'' = \pm 1$ we define the functionals $f_{m,i,j,\varepsilon',\varepsilon''}^+$ and $f_{m,i,j,\varepsilon',\varepsilon''}^- \in E_n$ as follows:

$$f_{m,i,j,\varepsilon',\varepsilon''}^+(x) = a\varepsilon' x_i + b\varepsilon''(x - i_{m,n}\pi_m(x))_j$$

and

$$f_{m,i,j,\varepsilon',\varepsilon''}^-(x) = f_{m,i,j,\varepsilon',\varepsilon''}^+(\mathrm{rot}_n(x))$$

where $\mathrm{rot}_n\colon E_n \to E_n$ is the isometry on E_n which changes the signs in the coordinates $\dim_2, \dim_3, \dots, \dim_n$ and leaves the other coordinates unchanged.

Let $\mathscr{F}_n^+$ and $\mathscr{F}_n^-$ be the set of these functionals, let $\dim_{n+1} = \dim_n + \mathrm{card}(\mathscr{F}_n^+) + 1$ and enumerate the elements of $\mathscr{F}_n^+$ and $\mathscr{F}_n^-$, correspondingly $g^+_{\dim_n+1}, \ldots, g^+_{\dim_{n+1}-1}$ and $g^-_{\dim_n+1}, \ldots, g^-_{\dim_{n+1}-1}$. Finally let $g^+_{\dim_{n+1}} = g^-_{\dim_{n+1}} = 0$.

The maps $i^+_{n,n+1}$ and $i^-_{n,n+1}$ are now defined as

$$i^+_{n,n+1}(x) = \left(x_1, \ldots, x_{\dim_n}, g^+_{\dim_n+1}(x), \ldots, g^+_{\dim_{n+1}-1}(x), 0\right)$$

and

$$i^-_{n,n+1}(x) = \left(x_1, \ldots, x_{\dim_n}, g^-_{\dim_n+1}(x), \ldots, g^-_{\dim_{n+1}-1}(x), 0\right).$$

Now make the crucial observation that these two injections are related as follows:

$$(*) \qquad i^-_{n,n+1} = \mathrm{rot}_{n+1} \circ i^+_{n,n+1} \circ \mathrm{rot}_n,$$

where $\mathrm{rot}_{n+1} = E_{n+1} \to E_{n+1}$ changes the signs of the coordinates $\dim_2, \ldots, \dim_{n+1}$ and leaves the others unchanged.

If $m < n$ we define $i^+_{m,n+1} = i^+_{n,n+1} i^+_{m,n}$ and $i^-_{m,n+1} = i^-_{n,n+1} i^-_{m,n}$ and note that the relation $(*)$ carries over

$$(**) \qquad i^-_{m,n+1} = \mathrm{rot}_{n+1} \circ i^+_{m,n+1} \circ \mathrm{rot}_m.$$

For the rest of the construction, we simply copy **[2]** (with the obvious modifications) to obtain two subspaces X^+ and X^- of l^∞ which have all the properties proved in **[2]**. Let C^+ and C^- be the unit-balls of X^+ and X^-, which are RN-sets.

But $C = C^+ + C^-$ contains an isometric copy of the ball of c_0 with radius 2. Indeed, let x_n be an element of E_n with $\|x_n\| \leqslant 1$, which has nonzero entries in the coordinates $\dim_2, \ldots, \dim_n$ at most. Then

$$2x_n = j_n^+(x_n) + j_n^-(x_n)$$

since the tails cancel out (the coordinates $\dim_{n+1}, \dim_{n+2}, \ldots$ of $j_n^+(x)$ and $j_n^-(x)$ are all zero while the other coordinates after $\dim_n$ cancel out because of $(**)$). Hence C contains all finite vectors supported by the coordinates $(\dim_n)_{n=1}^\infty$ and of norm less than or equal to 2; this finishes the presentation of our second example.

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