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NOTE ON H^2 ON PLANAR DOMAINS

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ABSTRACT. An analytic function $f\colon U \cup U' \to \mathbf{C}$, $U, U' \subset \mathbf{C}$ may belong to $H^2(U)$ and $H^2(U')$ and not to $H^2(U \cup U')$.

Let U be a domain in the complex plane and let $H^2(U)$ be the analytic functions f in U such that $|f|^2$ has a harmonic majorant in U. J. Conway asked the following question: Is it possible to have $f \in H^2(U)$, $f \in H^2(U')$ but $f \notin H^2(U \cup U')$? In this note we prove the following

THEOREM. *If $f\colon D \to \mathbf{C}$ is any analytic function on the unit disk D, then $D = U \cup U'$ where $f \in H^2(U)$ and $f \in H^2(U')$.*

We set some notation. $D_r = \{z \in \mathbf{C}\colon |z| < r\}$; $D = D_1$; $h(U, E, z)$ is the harmonic measure for the domain U, of the set $E \subset \partial U$ evaluated at $z \in U$. We will consider domains of the following type:

$$U = D_{1/2} \cup \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} T_n\right)$$

where A_n are annuli $\{z\colon a_n < |z| < b_n\}$ with $a_1 < \frac{1}{2}$, $a_n < b_n < a_{n+1}$, and $\lim_{n\to\infty} a_n = 1$; the T_n are tubes connecting A_n to A_{n+1}, $T_n = \{z\colon b_n \le |z| \le a_{n+1}$ and $|\arg z| < \pi\delta_n\}$. The numbers a_n, b_n, δ_n will be determined later. We write $U = U_{a_n, b_n, \delta_n}$. We now have two simple observations:

(1) $h(U, \partial U \cap \{z\colon |z| \ge r\}, 0) \le \delta_n$ provided $r > b_n$ and $b_n > \frac{1}{2}$;

(2) $h(U \cap D_{a_n},\ \partial(U \cap D_{a_n}) \cap \{z\colon |z| = a_n\}, 0) \le \delta_{n-1}$ provided $a_n > \frac{1}{2}$.

To see (1), observe that for Brownian motion started at 0 to exit U through $\partial U \cap \{z\colon |z| \ge r\}$, it must first get into the tube T_n and the probability of such paths is clearly $\le \delta_n$. (2) follows in the same way and we leave details to the reader.

LEMMA. *If $\phi\colon [0,1] \to \mathbf{R}^+$ is any (strictly) positive decreasing function, we can find $a_n, b_n, \delta_n, a'_n, b'_n, \delta'_n$ so that if $U = U_{a_n, b_n, \delta_n}$ and $U' = U_{a'_n, b'_n, \delta'_n}$, then*

(1) $D = U \cup U'$,

(2) $h(U \cap D_{a_n}, \partial(U \cap D_{a_n}) \cap \{z\colon |z| \le r\}, 0) \le \phi(r)$ provided there is an m such that $\frac{1}{2} < b_m < r$,

(3) same as (2) with U, a_n, b_n replaced by U', a'_n, b'_n, respectively.

PROOF. Choose a_n, b_n, a'_n, b'_n so that

$$D = D_{1/2} \cup \left(\bigcup_{n=1}^{\infty} \{z\colon a_n < |z| < b_n\}\right) \cup \left(\bigcup_{n=1}^{\infty} \{z\colon a'_n < |z| < b'_n\}\right).$$

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(1) is satisfied no matter what the choice of δ_n, δ'_n. So choose $\delta_n = \frac{1}{2}\phi(a_{n+2})$, $\delta'_n = \frac{1}{2}\phi(a'_{n+2})$. If r is given let $b_k = \max(b_j : b_j \leq r)$. It follows from observation (1) that

$$h(U, \partial U \cap \{z \colon |z| \geq r\}, 0) \leq \delta_k = \tfrac{1}{2}\phi(a_{k+1}) \leq \tfrac{1}{2}\phi(r).$$

From this and the maximum principle,

$$h(U \cap D_{a_n}, \partial(U \cap D_{a_n}) \cap \partial U \cap \{z \colon |z| \geq r\}, 0) \leq \tfrac{1}{2}\phi(r).$$

Since $\partial(U \cap D_{a_n}) \cap \partial U \cap \{z \colon |z| \geq r\} = \partial(U \cap D_{a_n}) \cap \{z \colon r \leq |z| < a_n\}$ we have

$$\begin{aligned} &h(U \cap D_{a_n}, \partial(U \cap D_{a_n}) \cap \{z \colon |z| \geq r\}, 0) \\ &\quad \leq \tfrac{1}{2}\phi(r) + h(U \cap D_{a_n}, \partial(U \cap D_{a_n}) \cap \{z \colon |z| = a_n\}, 0). \end{aligned}$$

By our observation, the last term is $\leq \delta_{n-1}$ and therefore $\leq \delta_k = \frac{1}{2}\phi(a_{k+2}) \leq \frac{1}{2}\phi(r)$. Putting this together gives (2). (3) is of course similar and the lemma is proved.

PROOF OF THE THEOREM. If $f \colon D \to \mathbf{C}$ is any analytic function (assumed unbounded) choose ϕ as above to satisfy $\int t\phi(r(t))\, dt < \infty$, where

$$r(t) = \min(r \colon \exists z \text{ with } |z| = r \text{ and } |f(z)| \geq t).$$

Construct U and U' as in the lemma using this ϕ. To see that $f \in H^2(U)$ consider the obvious harmonic majorant on $U \cap D_{a_n}$,

$$g_n(z) = \int_{\partial(U \cap D_{a_n})} |f|^2 \, dh(U \cap D_{a_n}, \cdot, z).$$

If $\lim_{n \to \infty} g_n$ exists (uniformly on compact sets) then it will be a harmonic majorant for $|f|^2$ on U. From Harnack's theorem, we only need to show $g_n(0)$ is bounded independently of n. Since

$$g_n(0) = 2 \int th(U \cap D_{a_n}, E_t, 0)\, dt$$

where $E_t = \{z \in U \cap D_{a_n} : |f(z)| \geq t\} \subset \{z \colon |z| \geq r(t)\}$, we have $h(U \cap D_{a_n}, E_t, 0) \leq \phi(r(t))$, by the lemma. This and the choice of ϕ imply $\{g_n(0)\}$ is bounded. The same argument shows that $f \in H^2(U')$ and the theorem is proved.

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