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PETTIS INTEGRABILITY AND THE EQUALITY OF THE NORMS OF THE WEAK* INTEGRAL AND THE DUNFORD INTEGRAL

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ABSTRACT. If (Ω, Σ, μ) is a perfect finite measure space and X is a Banach space, then it is shown that X^* has the μ-Pettis Integral Property if and only if

$$\left\|(\text{weak}^*)\text{-}\int_\Omega f\,d\mu\right\| = \left\|(\text{Dunford})\text{-}\int_\Omega f\,d\mu\right\|$$

for every bounded weakly measurable function $f\colon \Omega \to X^*$.

1. Introduction. Let X be a Banach space with dual X^* and (Ω, Σ, μ) a finite measure space. If $f\colon \Omega \to X^*$ is bounded and weakly measurable, that is if $x^{**} \circ f$ is measurable for every $x^{**} \in X^{**}$, then it can easily be shown that

(i) for every $E \in \Sigma$, there exists $x_E^* \in X^*$ such that, for every $x \in X$,

$$x_E^*(x) = \int_E \hat{x} \circ f\,d\mu$$

and

(ii) for every $E \in \Sigma$, there exists $x_E^{***} \in X^{***}$ such that, for every $x^{**} \in X^{**}$,

$$x_E^{***}(x^{**}) = \int_E x^{**} \circ f\,d\mu.$$

The element x_E^* is called the *weak* integral of f over E*, denoted by $(\mathrm{w}^*)\text{-}\int_E f\,d\mu$, and x_E^{***} is called the *Dunford integral of f over E*, denoted $(\mathrm{D})\text{-}\int_E f\,d\mu$. By definition, f is Pettis integrable if and only if $(\mathrm{D})\text{-}\int_E f\,d\mu \in X^*$.

A Banach space Y is said to have the *μ-Pettis Integral Property*, or μ-PIP if every bounded weakly measurable function $f\colon \Omega \to Y$ is Pettis integrable. Characterizations and properties of Pettis integrable functions, spaces with the μ-PIP, and integration of universally weakly measurable functions can be found in **[1, 5, 6, 8-14, 16]**. Clearly, X^* has the μ-PIP if and only if for every $f\colon \Omega \to X^*$ that is bounded and weakly measurable, $(\mathrm{w}^*)\text{-}\int_E f\,d\mu = (\mathrm{D})\text{-}\int_E f\,d\mu$ for every $E \in \Sigma$. We show that in fact if (Ω, Σ, μ) is perfect, then X^* has μ-PIP if and only if $\|(\mathrm{w}^*)\text{-}\int_\Omega f\,d\mu\| = \|(\mathrm{D})\text{-}\int_\Omega f\,d\mu\|$ for every such function f.

2. Preliminary results. If A is a finite subset of a Banach space X and $\varepsilon > 0$, then we define

$$C_{A,\varepsilon} = \left\{x^* \in B^*\colon |x^*(x)| < \varepsilon \text{ for every } x \in A\right\},$$

where $B^* = \{x^* \in X^*\colon \|x^*\| \leqslant 1\}$.

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LEMMA 1. *Let X be a Banach space and $x^{**} \in X^{**}$. Suppose for every $\eta > 0$ there exists a finite subset A of X^* and $\varepsilon > 0$ such that if $x^* \in C_{A,\varepsilon}$, then $|x^{**}(x^*)| < \eta$. Then $x^{**} \in X$.*

PROOF. Let x_α^* be a net in $\frac{1}{2}B^*$ such that x_α^* converges weak* to x^*. Then $(x_\alpha^* - x^*) \in B^*$ and is eventually in $C_{A,\varepsilon}$ for every A, ε. Hence, $x^{**}(x_\alpha - x^*)$ converges to zero. Consequently, x^{**} is weak* continuous on B^*, and hence, $x^{**} \in X$. $\square$

A function f from a measure space (Ω, Σ, μ) to a dual Banach space X^* is weak* measurable if $\hat{x} \circ f$ is measurable for every $x \in X$. If f is weak* measurable we define the *Pettis norm of* f by

$$\|f\|_P = \sup_{x \in B} \int_\Omega |\hat{x} \circ f| d\mu,$$

where B is the closed unit ball of X. If f is bounded, we also define the operator T_f: $X \to L^1$ by $T_f(x) = \hat{x} \circ f$ for $x \in X$. It is clear that the operator norm of T_f, $\|T_f\|_{\mathrm{OP}}$, is the same as the Pettis norm of f.

If $(f_\alpha)_{\alpha \in \Gamma}$ is a net of weak* measurable functions and if Σ_α is a sub σ-algebra of Σ for every $\alpha \in \Gamma$, then we say $(f_\alpha, \Sigma_\alpha)_{\alpha \in \Gamma}$ is a weak* martingale if

(i) $\Sigma_\alpha \subset \Sigma_\beta$ if $\alpha < B$,

(ii) for every $x \in X$, $E(\hat{x} \circ f_\beta | \Sigma_\alpha) = \hat{x} \circ f_\alpha$ if $\alpha < \beta$,

where $E(\cdot | \Sigma_\alpha)$ is the usual conditional expectation operator with respect to Σ_α (see [3]). This just says that $(\hat{x} \circ f_\alpha, \Sigma_\alpha)_{\alpha \in \Gamma}$ is a scalar-valued martingale for every $x \in X$.

We need the following rather deep result of Fremlin for perfect measure spaces [7]. (See [7, 15] for definitions and properties of perfect spaces.)

FREMLIN'S THEOREM. *Let (Ω, Σ, μ) be a finite perfect measure space and $(f_n)_{n=1}^\infty$ be a sequence of measurable extended real-valued functions on Ω. Then either $(f_n)_{n=1}^\infty$ has a subsequence which converges a.e. or $(f_n)_{n=1}^\infty$ has a subsequence having no measurable pointwise cluster points.*

We are now able to prove

PROPOSITION 2. *Let (Ω, Σ, μ) be a perfect measure space and X a Banach space. Suppose f: $\Omega \to X^*$ is bounded and weak* measurable. Then the following are equivalent*:

(i) *T_f: $X \to L^1(\mu)$ is a compact operator.*

(ii) *If $(f_\alpha, \Sigma_\alpha)_{\alpha \in \Gamma}$ is any bounded weak* martingale such that $\hat{x} \circ f_\alpha$ converges to $\hat{x} \circ f$ in $L^1(\mu)$ for every $x \in X$, then f_α converges to f in Pettis norm.*

(iii) *There exists a net of bounded simple functions $(f_\alpha)_{\alpha \in \Gamma}$, such that f_α converges to f in Pettis norm.*

PROOF. Without loss of generality f takes its range in B^*.

(i) $\Rightarrow$ (ii). Suppose T_f is compact and let $(f_\alpha, \Sigma_\alpha)_{\alpha \in \Gamma}$ be any bounded weak* martingale such that $\hat{x} \circ f_\alpha \to \hat{x} \circ f$ in $L^1(\mu)$. Letting T_{f_α}: $X \to L^1(\mu)$ by $T_{f_\alpha}(x) = \hat{x} \circ f_\alpha$, we note that T_{f_α} converges to $T_f(x)$ in $L^1(\mu)$ and $T_{f_\alpha}(x) = E(\hat{x} \circ f | \Sigma_\alpha)$. It suffices to show that $T_{f_\alpha}(x)$ converges to $T_f(x)$ uniformly on B, as this says

$$\lim_\alpha \sup_{x\in B} \left\|T_{f_\alpha}(x) - T_f(x)\right\|_{L^1} = 0, \text{ or}$$

$$\lim_\alpha \sup_{x\in B} \int_\Omega |\hat{x}\circ f_\alpha - \hat{x}\circ f|\,d\mu = 0.$$

Let $\varepsilon > 0$. Since T_f is compact, there exists $x_1,\ldots,x_n \in B$ such that $T_f(B) \subset \bigcup_{i=1}^n \{g: \|g - T_f(x_i)\|_{L^1} < \varepsilon/3\}$. Choose β such that if $\alpha > \beta$ then

$$\left\|T_{f_\alpha}(x_i) - T_f(x_i)\right\|_{L^1} < \varepsilon/3$$

for $i = 1,\ldots,n$. Let $x \in B$ and let x_i be such that $\|T_f(x) - T_f(x_i)\| < \varepsilon/3$. We note that $E(\cdot\,|\Sigma_\alpha)$ is an L^1 contraction [3]. Then if $\alpha > \beta$,

$$\begin{aligned}&\left\|T_{f_\alpha}(x) - T_f(x)\right\|_{L^1}\\ &\quad\leqslant \left\|T_{f_\alpha}(x) - T_{f_\alpha}(x_i)\right\|_{L^1} + \left\|T_{f_\alpha}(x_i) - T_f(x_i)\right\|_{L^1} + \left\|T_f(x_i) - T_f(x)\right\|_{L^1} < \varepsilon.\end{aligned}$$

(ii) $\Rightarrow$ (iii). It suffices to show that there exists a bounded weak* martingale $(f_\alpha, \Sigma_\alpha)_{\alpha\in\Gamma}$ such that f_α is simple for every $\alpha \in \Gamma$, and $\hat{x}\circ f_\alpha \to \hat{x}\circ f$ in L^1 for every $x \in X$.

Let Π be the set of finite partitions of Ω into elements of Σ directed by refinement. If $\pi \in \Pi$, let Σ_π be the finite σ-algebra generated by the elements of π and let

$$f_\pi = \sum_{A\in\pi} \frac{(\mathrm{w}^*)\text{-}\int_A f\,d\mu}{\mu(A)}\chi_A.$$

It is clear that $(f_\pi, \Sigma_\pi)_{\pi\in\Pi}$ is a weak* martingale, each f_π is simple and the fact that $\hat{x}\circ f_\pi \to \hat{x}\circ f$ in $L^1(\mu)$ follows from scalar-valued martingale convergence theorems [3].

(iii) $\Rightarrow$ (i). Suppose $(f_\alpha)_{\alpha\in\Gamma}$ is a net of simple functions converging to f in Pettis norm. Then T_{f_α} converges to T_f in operator norm. Since T_{f_α} is a finite rank operator for each α, T_f is compact. □

The following was first observed by Stegall [8].

PROPOSITION 3. *If (Ω, Σ, μ) is a perfect finite measure space and f: $\Omega \to X^*$ is bounded and weakly measurable, then T_f: $X \to L^1$ is compact. Hence there exists a net of simple functions converging to f in Pettis norm.*

PROOF. Let $(x_n)_{n=1}^\infty$ be bounded in X. Suppose $(\hat{x}_n\circ f)_{n=1}^\infty$ does not have an a.e. convergent subsequence. By Fremlin's theorem, there is a subsequence $(\hat{x}_{n_j}\circ f)_{j=1}^\infty$ having no measurable pointwise cluster points. Let x^{**} be a weak* cluster point of $(\hat{x}_{n_j})_{j=1}^\infty$ in X^{**}. Hence $x^{**}\circ f$ is a pointwise cluster point of $(\hat{x}_{n_j}\circ f)_{j=1}^\infty$ and is therefore nonmeasurable. This contradicts the weak measurability of f. Hence some subsequence must converge a.e. and by boundedness this subsequence must converge in $L^1(\mu)$. □

3. Main result. Putting together the pieces from §2 yields

THEOREM 4. *If* (Ω, Σ, μ) *is a perfect measure space and* X *is a Banach space, then* X^* *has* μ*-PIP if and only if for every* $f\colon \Omega \to X^*$ *that is bounded and weakly measurable*

$$\left\|(\mathrm{w}^*)\text{-}\int_\Omega f\,d\mu\right\| = \left\|(\mathrm{D})\text{-}\int_\Omega f\,d\mu\right\|.$$

In fact if f *is not Pettis integrable, then for some* $E \in \Sigma$ *and* $\alpha > 0$ *there exists a sequence of simple functions* $(f_n)_{n=1}^\infty$ *such that*

$$\left\|(\mathrm{w}^*)\text{-}\int_E f - f_n\,d\mu\right\| \to 0 \quad \textit{but} \left\|(\mathrm{D})\text{-}\int_E f - f_n\,d\mu\right\| > \alpha \quad \textit{for every } n.$$

PROOF. Of course if f is Pettis integrable then these two norms are the same. Conversely, let $f\colon \Omega \to X^*$ be bounded and weakly measurable. Without loss of generality, f takes its range in B^*. By Lemma 1, it suffices to show that for every $E \in \Sigma$ and $\eta > 0$ there exists a finite subset A of X^* and $\varepsilon > 0$ such that $|\int_E x^{**} \circ f\,d\mu| < \eta$ whenever $x^{**} \in C_{A,\varepsilon}$.

Let $E \in \Sigma$ and $\eta > 0$. Let $f_E = f\chi_E$. Choose by Proposition 3 a simple function h such that $\|f_E - h\|_P < n/2$. Note then by hypothesis

$$\left\|(\mathrm{D})\text{-}\int_\Omega (f_E - h)\,d\mu\right\| = \left\|(\mathrm{w}^*)\text{-}\int_\Omega (f_E - h)\,d\mu\right\| \leqslant \|f_E - h\|_P < \eta/2.$$

Hence letting A be the range of h and $\varepsilon = \eta/2$, we see that if $x^{**} \in C_{A,\varepsilon}$, then

$$\begin{aligned}\left|\int_E x^{**} \circ f\,d\mu\right| &\leqslant \left|\int_\Omega x^{**} \circ (f_E - h)\,d\mu\right| + \left|\int_\Omega x^{**} \circ h\,d\mu\right| \\ &< \left\|(\mathrm{D})\text{-}\int_\Omega (f_E - h)\,d\mu\right\| + \eta/2 < \eta.\end{aligned}$$

This proves the first assertion. To prove the second, we know that if f is bounded and weakly measurable, we can always find a sequence of simple functions (f_n) such that $\|(\mathrm{w}^*)\text{-}\int_\Omega (f - f_n)\|$ converges to zero. If there did not exist an $\alpha > 0$ such that $\|(\mathrm{D})\text{-}\int_\Omega (f - f_n)\| > \alpha$, then this would force f to be Pettis integrable as in the above argument. □

The following example is due to Phillips and is discussed in detail by Geitz in [**9** and **10**]. Let (Ω, Σ, μ) be usual Lebesgue measure space and $l^\infty[0,1]$ be the space of bounded functions with usual supremum norm. Sierpiński constructed a subset B of $[0,1] \times [0,1]$ such that

(i) for every $t_0 \in [0,1]$, $\{s\colon (s, t_0) \in B\}$ is countable, and

(ii) for every $s_0 \in [0,1]$, $\{t\colon (s_0, t) \notin B\}$ is countable.

It is shown in [**10**] that the function $f\colon [0,1] \to l^\infty[0,1]$ given by $[f(s)](t) = \chi_B(s,t)$ is bounded and weakly measurable but not Pettis integrable with respect to (Ω, Σ, μ).

It is also shown that if e_{t_0} is the evaluation functional at t_0 on $l^\infty[0,1]$, then from (i) we have

$$\int_{[0,1]} e_{t_0} f(s)\,d\mu(s) = \int_{[0,1]} \chi_B(s, t_0)\,d\mu(s) = 0.$$

Hence, $\|(\mathrm{w}^*)\text{-}\int_E f\,d\mu\| = 0$ for every $E \in \Sigma$. However, if $\beta \in \mathrm{ba}[0,1] = (l^\infty[0,1])^*$ is such that β vanishes on countable sets and $\|\beta\| = 1$, then we have by (ii) that, for every $s_0 \in [0,1]$,

$$\int_{[0,1]} f(s_0)\,d\beta = 1.$$

Hence, $\int_E \int_{[0,1]} f(s)\,d\beta\,d\mu(s) = \mu(E)$ for every $E \in \Sigma$, and thus $\|(\mathrm{D})\text{-}\int_E f\,d\mu\| = \mu(E)$ for every $E \in \Sigma$.

4. Observations and questions. Suppose $f\colon \Omega \to X^*$ is bounded and weak* measurable, that is $\hat{x} \circ f$ is measurable for every $x \in X$, such that T_f is weak compact. Hence, $T_f^{**}\colon X^{**} \to L^1(\mu)$. This will certainly, but not necessarily, be the case if f is weakly measurable. Since $\langle T_f^{**}(x^{**}), g\rangle = x^{**}((\mathrm{w}^*)\text{-}\int_\Omega fg)$ for every $g \in L^\infty(\mu)$, there exists a function $h_{x^{**}} \in L^1(\mu)$ such that $x^{**}((\mathrm{w}^*)\text{-}\int_\Omega fg\,d\mu) = \int_\Omega h_{x^{**}} g\,d\mu$ for every $g \in L^\infty(\mu)$. Note that if (x_β) is a net in X such that $\hat{x}_\beta$ converges weak* to x^{**}, then clearly $x^{**} \circ f$ is a pointwise limit of $(\hat{x}_\beta \circ f)$, whereas $h_{x^{**}}$ is an $L^1(\mu)$ limit of $(\hat{x}_\beta \circ f)$. Consequently, for every x^{**} there exists a sequence $(x_n)_{n=1}^\infty$ in X such that $\hat{x}_n \circ f$ converges a.e. to $h_{x^{**}}$. If f is Pettis integrable, however, it must be the case that $x^{**} \circ f = h_{x^{**}}$ a.e. Hence we get

THEOREM 5. *Let X be a Banach space and (Ω, Σ, μ) a finite measure space. Suppose $f\colon \Omega \to X^*$ is bounded and weakly measurable. Then f is Pettis integrable if and only if, for every $x^{**} \in X^{**}$, there exists a bounded sequence $(x_n)_{n=1}^\infty$ in X such that both of the following hold*:

(i) *$\hat{x}_n \circ f$ converges a.e. to $x^{**} \circ f$,*

(ii) *$\hat{x}_n((\mathrm{w}^*)\text{-}\int_E f\,d\mu)$ converges to $x^{**}((\mathrm{w}^*)\text{-}\int_E f\,d\mu)$ for every $E \in \Sigma$.*

We observe that condition (i) above guarantees that $\|(\mathrm{w}^*)\text{-}\int f\,d\mu\| = \|(\mathrm{D})\text{-}\int f\,d\mu\|$. Hence we get the following corollary from Theorems 4 and 5:

COROLLARY 6. *Let X be a Banach space and (Ω, Σ, μ) a perfect finite measure space. Then X^* has μ-PIP if and only if whenever $f\colon \Omega \to X^*$ is bounded and weakly measurable then, for every $x^{**} \in X^{**}$, there is a bounded sequence $(x_n)_{n=1}^\infty$ in X such that $\hat{x}_n \circ f$ converges to $x^{**} \circ f$ a.e.*

Question. Is it possible to remove condition (ii) in Theorem 5?

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