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A HÖRMANDER TYPE CRITERION FOR QUASI-RADIAL FOURIER MULTIPLIERS

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ABSTRACT. We state practicable sufficient conditions on quasi-radial functions $m \circ \rho(\xi) = m(\rho(\xi))$ to be Fourier multipliers in $L^p(\mathbf{R}^n)$. Here m is a bounded function and ρ is a homogeneous distance function. The conditions on m are given in terms of localized Bessel potentials and those on ρ reflect and generalize basic properties of the norm in $\mathbf{R}^n$. The results are related to those of Madych [7] and Fabes and Rivière [3] and improve their results (specialized to quasi-radial multipliers). The proof utilizes Madych's approach [7] and interpolation properties of localized Bessel potential spaces [2].

0. A function $m \in L^\infty(\mathbf{R}^n)$ is called a Fourier multiplier in $L^p(\mathbf{R}^n)$, $m \in M_p(\mathbf{R}^n)$, if the relation $(Tf)\hat{}(\xi) = m(\xi)\hat{f}(\xi)$ defines a continuous endomorphism on $L^p(\mathbf{R}^n)$ where $\hat{f}$ denotes the Fourier transform of $f \in L^p \cap L^2(\mathbf{R}^n)$; the multiplier norm $\|m\|_{M_p}$ equals the operator norm of T. For the general definition of localized Bessel potential spaces $S(q, \gamma)$ we refer to Connett and Schwartz [2]. In case $\gamma > 1/q$, $1 < q < \infty$, these spaces can be identified with the spaces of functions of weak bounded variation, as was shown by Gasper and Trebels [5]. If $\gamma \in \mathbf{N}$, $1 \leqslant q \leqslant \infty$, then $\mathrm{WBV}_{q,\gamma} = \{m \in L^\infty(0, \infty)\colon m, m', \ldots, m^{(\gamma-1)}$ are locally absolutely continuous and $\|m\|_{q,\gamma} = \|m\|_\infty + \sup_{R>0}(\int_R^{2R} |t^\gamma m^{(\gamma)}(t)|^q\, dt/t)^{1/q} < \infty$ if $q < \infty$ or $\|m\|_{q,\gamma} = \|m\|_\infty + \operatorname{ess\,sup}_{t>0}|t^\gamma m^{(\gamma)}(t)| < \infty$ if $q = \infty\}$. A function ρ is called a (P-)homogeneous distance function if ρ is continuous on $\mathbf{R}^n$, positive on $\mathbf{R}_0^n = \mathbf{R}^n \setminus \{0\}$ and satisfies $\rho(t^P x) = t\rho(x)$ for $t > 0$ and $x \in \mathbf{R}^n$. Here P is a fixed real $(n \times n)$-matrix with eigenvalues having positive real parts.

Examples are given in [**1, 10** and **14**]; further examples are:

$$\rho_1(x) = x_1^6 - \sin^2(x_1^3/x_2)x_1^3x_2 + x_2^2, \qquad \rho_2(x) = \sum_{j=1}^n |x_j|^{\alpha_j} \qquad (\alpha_j > 0),$$

$$\rho_3(x) = (x_1^4 + x_2^2)^{1/2} + |x_1|^{4/5}|x_2|^{8/5}/(x_1^2 + |x_2|).$$

Throughout the paper c and C denote generic constants, N the least integer greater than $n/2$, and $t_+ = \max\{0, t\}$.

THEOREM 1. *Let $\rho \in C^N(\mathbf{R}_0^n)$ be a homogeneous distance function, $1 < p < \infty$, and $m \in \mathrm{WBV}_{q,\gamma}$, where $1 \leqslant q \leqslant \infty$ and $\gamma > n|1/p - 1/2| + (1/q - |1/p - 1/2|)_+$. Then $m \circ \rho \in M_p(\mathbf{R}^n)$ and $\|m \circ \rho\|_{M_p} \leqslant C\|m\|_{q,\gamma}$ with C independent of m.*

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Peral and Torchinsky [9] have shown: if $0 < p < \infty$, $1 < q \leqslant \infty$ and $m \in \mathrm{WBV}_{q,\gamma}$ for some integer $\gamma > \nu|1/p - 1/2| + 1/q$, then $m \circ \rho$ is a multiplier in $H^p(\mathbf{R}^n)$. Here $\rho \in C^\infty(\mathbf{R}^n_0)$ is the P-homogeneous distance function determined by $|\rho(x)^{-P}x| = 1$ if $x \neq 0$. The matrix P has to satisfy $t^\alpha|x| \leqslant |t^P x| \leqslant t^\beta|x|$ for some $1 \leqslant \alpha \leqslant \beta$ and all $t \geqslant 1$, $x \in \mathbf{R}^n$. ν is the trace of P.

Obviously Theorem 1 is an improvement of Peral and Torchinsky's result in case $1 < p < \infty$ as the following examples show:

(i) $(1 - \rho(\xi))_+^\lambda \in M_p(\mathbf{R}^n)$ if $\lambda > (n-1)|1/p - 1/2|$ and $1 < p < \infty$. This multiplier may be of interest for the generalized Riesz summation of inverse Fourier integrals. Analogous results were obtained by Löfström [6] and Peetre [8] for a class of distance functions ρ satisfying $\rho \in C^\infty(\mathbf{R}^n_0)$ and $\rho(tx) = t^\alpha\rho(x)$ for some $\alpha > 0$ and all $t > 0$, $x \in \mathbf{R}^n$. See also the discussion of the above multiplier in the radial case in Fefferman [4].

(ii) $e^{i\rho(\xi)}(1 + \rho(\xi))^{-\beta} \in M_p(\mathbf{R}^n)$ if $\beta > n|1/p - 1/2|$ and $1 < p < \infty$. This result extends that of Sjöstrand [11] from the radial to the quasi-radial case.

For the proof of Theorem 1 the approach via Littlewood-Paley functions is used. We employ the same Littlewood-Paley functions as Madych [7]:

$$g_1(f)(x) = \left(\int_0^\infty |K_t * K_t * f(x)|^2\, dt/t\right)^{1/2},$$

$$g_2(f)(x) = \left(\int_0^\infty \int t^{-\nu}\left(1 + |t^{-P^*}(x-y)|^\gamma\right)^{-2} |K_t * f(y)|^2\, dy\, dt/t\right)^{1/2},$$

where $K_t = F^{-1}[\varphi(t\, r(\xi))]$, $\varphi \in C^\infty(\mathbf{R})$ is a nonnegative bump function supported in $[1, 2]$, $r \in C^\infty(\mathbf{R}^n_0)$ is a fixed P-homogeneous distance function (see [14, p. 1255]), F^{-1} denotes the inverse Fourier transformation, P^* the transpose of P, $\nu = \mathrm{trace}(P)$, $f \in L^2(\mathbf{R}^n)$ and $\gamma > n/2$.

To be precise we point out that the matrix P is more general than that of Madych [7] which is assumed to be "good" or at least "reasonable", but this does not affect the L^p-behaviour of the g-functions.

The approach via Littlewood-Paley functions first occurs in Stein [12] and is generalized by Madych [7] to obtain an anisotropic version (Proposition 4) of the Hörmander multiplier criterion [13]. Madych's assumption on the multiplier m can be expressed by

$$\sup_{t>0} \int \left| |x|^\gamma F^{-1}\left[\varphi(r(\xi))m(t^P\xi)\right](x)\right|^2 dx \leqslant B^2 < \infty, \qquad \gamma > n/2.$$

The following lemma presents a practicable sufficient condition for this assumption to be satisfied in the quasi-radial case.

LEMMA 1. *Let $m \in C^\infty(0,\infty)$ be compactly supported in $(0,\infty)$ and let ρ be as in Theorem* 1. *Then for every $\gamma > n/2$*

$$\int \left| |x|^\gamma F^{-1}[\varphi(r(\xi))m(t\rho(\xi))](x)\right|^2 dx \leqslant c\|m\|_{2,\gamma}^2$$

with c independent of m and $t > 0$.

Madych's method now yields

LEMMA 2. *Let m, ρ, and γ be as in Lemma* 1. *Then $\|m \circ \rho\|_{M_p} \leqslant c\|m\|_{2,\gamma}$ for every $1 < p < \infty$, where c is independent of m.*

A combination of Lemma 2 with interpolation and embedding arguments of Connett and Schwartz [**2**] on localized Bessel potentials will lead to Theorem 1.

1.

PROOF OF LEMMA 1. (a) *The case $\gamma = k \in \mathbf{N}$.* We omit the region of integration if it is the whole $\mathbf{R}^n$. $\sigma \in \mathbf{N}^n$ is a multi-index, $|\sigma| = \sigma_1 + \cdots + \sigma_n$, $x^\sigma = x_1^{\sigma_1} \cdots x_n^{\sigma_n}$, $D^\sigma = (\partial/\partial x)^\sigma$, $(\partial/\partial x)^\sigma = (\partial/\partial x_1)^{\sigma_1} \cdots (\partial/\partial x_n)^{\sigma_n}$. By Plancherel's theorem and simple estimates,

$$\begin{aligned} I_k &= \int \left| |x|^k F^{-1}[m(R\rho(\cdot))\varphi(r(\cdot))](x)\right|^2 dx \\ &\leqslant c \sum_{|\sigma|=k} \int \left|\left(\frac{\partial}{\partial\xi}\right)^\sigma [m(R\rho(\xi))\varphi(r(\xi))]\right|^2 d\xi \\ &\leqslant c \sum_{|\sigma|\leqslant k} \int_{1\leqslant r(\xi)\leqslant 2} \left|\left(\frac{\partial}{\partial\xi}\right)^\sigma [m(R\rho(\xi))]\right|^2 d\xi. \end{aligned} \tag{1.1}$$

It is an immediate consequence of ρ and r being P-homogeneous distance functions that there are $c_1, c_2 > 0$ such that $c_1 r(\xi) \leqslant \rho(\xi) \leqslant c_2 r(\xi)$. Hence, the domain of integration is contained in $\{\xi\colon c_1 \leqslant \rho(\xi) \leqslant 2c_2\}$ and we may estimate the derivatives and arbitrary powers of ρ from above and below.

Applying the chain rule and converting to Riemann-Stieltjes integrals we get

$$\begin{aligned} I_k &\leqslant c \sum_{l=0}^{k} \int_{c_1\leqslant\rho(\xi)\leqslant 2c_2} \left|(R\rho(\xi))^l m^{(l)}(R\rho(\xi))\right|^2 \rho(\xi)^{-\nu} d\xi \\ &\leqslant c \sum_{l=0}^{k} \int_{c_1 R}^{2c_2 R} \left|t^l m^{(l)}(t)\right|^2 dt/t \leqslant c\|m\|_{2,k}^2 \end{aligned}$$

since we can cover $[c_1 R, 2c_2 R]$ with a fixed number of intervals $[2^j, 2^{j+1}]$ and $\mathrm{WBV}_{2,l} \subset \mathrm{WBV}_{2,k}$ for $l \geqslant k$.

(b) *The case $\gamma = k + \delta$, $0 < \delta < 1$.* This case can be treated by the argument of [**2**] on pp. 70–71. Hence, we only point out the important steps. Let $S(q,\alpha)$ be defined as a V^*-localization of the Bessel potential space L_α^q, and $S_0(q,\alpha)$ as a W^*-localization. For the definitions we refer to [**2**, §§1.3 and 2.4].

Let the space $\mathscr{F} = \mathscr{F}(S_0(2, k), S_0(2, k+1))$ of functions on $\bar{S} = \{z \in \mathbf{C}: 0 \leqslant \operatorname{Re} z \leqslant 1\}$ be defined as in [**2**, §1.4]. In order to smooth functions in $S_0(q, \alpha)$ take $\eta \in C^\infty(0, \infty)$ with $\operatorname{supp} \eta = [1/2, 2]$, $\eta \geqslant 0$ and $\int_0^\infty \eta(t)\, dt/t = 1$, set $\eta_\varepsilon(t) := \varepsilon\eta(t^\varepsilon)$ for $\varepsilon > 0$ and define for any $f \in \mathscr{F}$ with $f(\delta) = m$

$$f_\varepsilon(z, t) := \int_0^\infty \eta_\varepsilon(t/s) f(z, s)\, ds/s,$$

$f(z, s)$ being the value of $f(z) \in S_0(2, k)$ at s. Clearly

$$m_\varepsilon(t) = \int_0^\infty \eta_\varepsilon(t/s) m(s)\, ds/s = f_\varepsilon(\delta, t).$$

We apply the Phragmen-Lindelöf principle to the function $z \to G_z f_\varepsilon$ with

$$G_z f_\varepsilon(x) = |x|^{k+z} \int \varphi(r(\xi)) f_\varepsilon(z, R\rho(\xi)) e^{ix\xi}\, d\xi.$$

Observe that analogously to (a)

$$\begin{aligned} \|G_z f_\varepsilon\|_2^2 &\leqslant \int \left| \left(1 + |x|^{k+1}\right) \int \varphi(r(\xi)) f_\varepsilon(z, R\rho(\xi)) e^{ix\xi}\, d\xi \right|^2 dx \\ &\leqslant c\|f_\varepsilon(z)\|_{2, k+1}^2 \leqslant c(1+\varepsilon)^2 \|f(z)\|_{2, k}^2. \end{aligned}$$

For the last step we used the integration by parts and $\|m_\varepsilon\|_{2, l} \leqslant C\|m\|_{2, l}$. Hence, for fixed $\varepsilon > 0$ the function $z \to G_z f_\varepsilon$ is L^2-valued and uniformly bounded for $z \in \bar{S}$.

Since f is continuous on $\bar{S}$ and analytic in the interior of $\bar{S}$, one can show by estimates similar to those above that $z \to G_z f_\varepsilon$ is continuous and analytic with respect to the L^2-norm.

On the boundary for $j = 0, 1$

$$\|G_{j+iy} f_\varepsilon\|_2 \leqslant c\|f_\varepsilon(j+iy)\|_{2, k+j} \leqslant c\|f(j+iy)\|_{2, k+j}.$$

Hence,

$$\|G_z f_\varepsilon\|_2 \leqslant c \max_{j=0,1} \max_{y \in \mathbf{R}} \|f(j+iy)\|_{2, k+j} = c\|f\|_{\mathscr{F}},$$

c independent of ε. By Lebesgue's Dominated Convergence Theorem and Fatou's Lemma

$$\begin{aligned} &\int \left| |x|^\gamma \int \varphi(r(\xi)) m(R\rho(\xi)) e^{ix\xi}\, d\xi \right|^2 dx \\ &\quad \leqslant \liminf_{\varepsilon \to \infty} \int \left| |x|^\gamma \int \varphi(r(\xi)) m_\varepsilon(R\rho(\xi)) e^{ix\xi}\, d\xi \right|^2 dx \\ &\quad = \liminf_{\varepsilon \to \infty} \|G_\delta f_\varepsilon\|_2^2 \leqslant c\|f\|_{\mathscr{F}}^2. \end{aligned}$$

The interpolation property then gives the desired result since the norm of m as an element of the complex interpolation space

$$[S_0(2, k), S_0(2, k+1)]_\delta = S_0(2, \gamma)$$

is equal to $\inf\{\|f\|_{\mathscr{F}}: f \in \mathscr{F}, f(\delta) = m\}$.

PROOF OF THEOREM 1. The restriction in Lemma 1 on m to be infinitely differentiable and compactly supported away from the origin can be removed by density arguments as in [**13**, 6.2.1 and **2**, p. 25]. On account of Lemma 2 the bilinear operator B defined by $(B(m,f))\hat{}(\xi) = m(\rho(\xi))f\hat{}(\xi)$ maps $S(q_i,\gamma_i) \oplus L^{p_i} \to L^{p_i}$ continuously for $q_1 = 2$, $\gamma_1 = n/2 + \varepsilon$, $p_1 = 1 + \varepsilon$, any small $0 < \varepsilon < 1/2$. Taking into account the embeddings $S(q,\alpha) \subset S(2,\alpha)$ if $2 \leqslant q \leqslant \infty$ and $S(q,\alpha + 1/2) \subset S(2,\alpha)$ if $1 < q < 2$ [**2**, Theorem 5.3], we deduce that B is also continuous for $q_2 = 1/\varepsilon$, $\gamma_2 = \gamma_1$, $p_2 = p_1$ and $q_3 = 1 + \varepsilon$, $\gamma_3 = (n+1)/2 + \varepsilon$, $p_3 = p_1$. On the other hand $S(q,\alpha) \subset L^\infty(0,\infty)$ if $1 < q < \infty$ and $\alpha > 1/q$ [**2**, Theorem 5.4] and $M_2(\mathbf{R}^n) = L^\infty(\mathbf{R}^n)$, hence B is continuous for $q_4 = 1/\varepsilon$, $\gamma_4 = 2\varepsilon$, $p_4 = 2$ and $q_5 = 1 + \varepsilon$, $\gamma_5 = 1$, $p_5 = 2$. The range R in the $(1/q,\gamma,1/p)$-diagram for which B is continuous thus contains the five points $(1/q_i,\gamma_i,1/p_i)$, $i = 1,\dots,5$, and by duality also the three points $(1/q_i,\gamma_i,1 - 1/p_i)$, $i = 1,2,3$. By Connett and Schwartz [**2**, Theorem 5.5] R is convex and since we may take $\varepsilon > 0$ arbitrarily small R contains the interior of the convex hull H of the eight points $(1/2, n/2, 1)$, $(1/2, n/2, 0)$, $(0, n/2, 1)$, $(0, n/2, 0)$, $(1, (n+1)/2, 1)$, $(1, (n+1)/2, 0)$, $(0, 0, 1/2)$, $(1, 1, 1/2)$. Note $S(q,\beta) \subset S(q,\alpha)$ for $\beta \geqslant \alpha$, thus for every $(1/q,\alpha,1/p) \in H$ also the line $\{(1/q,\beta,1/p)\colon \beta \geqslant \alpha\}$ belongs to R. The region so obtained can be described by the two pairs of inequalities $\gamma > (n-1)|1/p - 1/2| + 1/q$ if $1/q \geqslant |1/p - 1/2|$ or $\gamma > n|1/p - 1/2|$ if $1/q \leqslant |1/p - 1/2|$, $1 < p, q < \infty$. Finally, in the context of WBV-spaces, we can admit $q = 1$ and $q = \infty$ without leaving R since $\mathrm{WBV}_{1,\alpha+\varepsilon} \subset \mathrm{WBV}_{1+\varepsilon,\alpha} = S(1+\varepsilon,\alpha)$ if $\alpha \geqslant 1$, $\varepsilon > 0$ and $\mathrm{WBV}_{\infty,\alpha} \subset \mathrm{WBV}_{q,\alpha} = S(q,\alpha)$ if $1 < q < \infty$, $\alpha > 1/q$. (See [**5**, Theorems 3 and 4].) This completes the proof of the theorem.

2. The above method of proof can be employed to give the following variant of Theorem 1:

THEOREM 2. *Let ρ be a P-homogeneous distance function with $P = \operatorname{diag}(\alpha_1,\dots,\alpha_n)$. For each $j = 1,\dots,n$ and $\sigma \in \{0,1\}^n$ with $\sigma_j = 0$ let $x_j \to D^\sigma\rho(x)$ be locally absolutely continuous for almost all $x^j = (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n) \in \mathbf{R}^{n-1}$. For $\sigma^1,\dots,\sigma^k \in \{0,1\}^n$ such that $\sigma^1 + \cdots + \sigma^k \in \{0,1\}^n$ let $\prod_{j=1}^k D^{\sigma^j}\rho \in L^s_{\mathrm{loc}}(\mathbf{R}^n_0)$ for some s with $2 < s \leqslant \infty$. Now, if $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ and $m \in \mathrm{WBV}_{q,\gamma}$ for some*

$$\gamma > n|1/p - 1/2| + \big(1/q - (1 - 1/s)|1/p - 1/2|\big)_+,$$

then $m \circ \rho \in M_p(\mathbf{R}^n)$ and $\|m \circ \rho\|_{M_p} \leqslant c\|m\|_{q,\gamma}$ with c independent of m.

This result cannot be obtained from the general results of Madych [**7**], Fabes and Rivière [**3**], and Peral and Torchinsky [**9**]. It allows one to deal with nonsmooth distance functions ρ, e.g.

$$\rho_4(x) = |x_1| + \big(x_1^2 + |x_2|\big)^{1/2},$$

$$\rho_5(x) = \sum_{j=1}^{3} |x_j|^{\alpha_j} + \tfrac{1}{2}\operatorname{sgn}(x_1)|x_1|^{\beta_1}|x_2|^{\beta_2}, \qquad \alpha_j, \beta_k > \tfrac{1}{2}, \beta_1/\alpha_1 + \beta_2/\alpha_2 = 1.$$

Application of Theorem 2 to $\rho = \rho_2,\dots,\rho_5$ yields

(i) $(1 - \rho(\xi))_+^\lambda \in M_p(\mathbf{R}^n)$ if $\lambda > (n - 1 + 1/s)|1/p - 1/2|$ and $1 < p < \infty$.

In the different cases it suffices if $s > 2$ satisfies $1/s > \max(1 - \alpha_j)$ for ρ_2, $s < 5$ for ρ_3, $s = \infty$ for ρ_4, and $1/s > \max(1 - \alpha_j)$ as well as $1/s > \max(1 - \beta_k)$ for ρ_5.

(ii) $e^{i\rho(\xi)}(1 + \rho(\xi))^{-\beta} \in M_p(\mathbf{R}^n)$ if $\beta > n|1/p - 1/2|$ and $1 < p < \infty$.

The proof of Theorem 2 is analogous to that of Theorem 1, therefore we only indicate the modifications: Instead of g_2 one considers

$$g_3(f)(x) = \left(\int_0^\infty \int t^{-\nu} \left(\prod_{j=1}^n \left(1 + t^{-\alpha_j}|x_j - y_j|\right)\right)^{-2\kappa} |K_t * f(y)|^2 \, dy \, dt/t \right)^{1/2},$$

where $\kappa > 1/2$. This g-function has the same L^p-behaviour as the g_2-function. Instead of Lemma 1 one proves

$$\int \left| \left(\prod_{j=1}^n \left(1 + |x_j|\right)\right)^{\kappa} F^{-1}[\varphi(r(\xi))m(t\rho(\xi))](x) \right|^2 dx \leqslant c\|m\|_{q,\gamma}^2$$

if $\gamma = n\kappa$, $1/2 < \kappa \leqslant 1$, $1/q = 1/2 - \kappa/s$. This can be done in the same manner as in the proof of Lemma 1 but interpolating between $\gamma = 0$ and $\gamma = n$. In case $\gamma = n$ one has to apply Hölder's inequality to (1.1) to separate the terms depending on m from those containing ρ and its partial derivatives.

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