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STEINER MINIMAL TREE FOR POINTS ON A CIRCLE

D. Z. DU, F. K. HWANG AND S. C. CHAO

ABSTRACT. We show that the Steiner minimal tree for a set of points on a circle is the shortest path connecting them if at most one distance between two consecutive points is "large". We prove this by making an interesting use of the Steiner ratio ρ which has been well studied in the literature.

1. Introduction. Let A denote a given set of n points in the Euclidean plane. A *Steiner minimal tree* for A is the shortest network (clearly, it has to be a tree) interconnecting A. Junctions of the network which are not in A are called *Steiner points.* A tree connecting A without using any Steiner points is called a *spanning tree* and a shortest spanning tree is called a *minimal spanning tree.* Let $L_S(A)$ and $L_M(A)$ denote the lengths of a Steiner minimal tree and a minimal spanning tree for A, respectively. Define the *Steiner ratio* $\rho = \inf_A L_S(A)/L_M(A)$. Gilbert and Pollak [**4**] conjectured that $\rho = \sqrt{3}/2$ and it has recently been proved [**2**] that $\rho \geq .8$.

While efficient algorithms [**6, 7**] exist for constructing minimal spanning trees, the problem of constructing Steiner minimal trees for general set of points is known [**3**] to be NP-complete. However, efficient algorithms may exist for point-sets with special structures. In this paper we study the case that A is a set of points on a circle with at most one "large" distance between two consecutive points. We prove that a minimal spanning tree for A (the tree turns out to be a path) is also a Steiner minimal tree for A by making an interesting use of the Steiner ratio.

2. Some preliminary results. A path $B_1B_2\cdots B_n$ will be called a *Steiner path* if it is convex and all subtending angles are 120°. Let $d(x,y)$ denote the distance between two points x and y. The following two lemmas can be easily verified by elementary geometry. We state them without proofs.

LEMMA 1. *Suppose that $B_1B_2\cdots B_n$, $n \leq 1$, is a Steiner path such that $d(B_i, B_{i+1}) \leq a$ for $i = 1, \ldots, n-1$. Then $d(B_1, B_n) \leq 2a$.*

LEMMA 2. *Let W be any point in the triangle $\triangle XYZ$. Then*

$$d(X,W) \leq \max\{d(X,Y), d(X,Z)\}.$$

LEMMA 3. *A necessary and sufficient condition for the path $B_1B_2\cdots B_n$ to be a minimal spanning tree is that*

$$d(B_i, B_k) \geq d(B_j, B_{j+1}) \quad \textit{for all } 1 \leq i \leq j < k \leq n.$$

PROOF. The proof follows immediately from a well-known [**1**] condition for a tree to be a minimal spanning tree. It is also easily seen if all inequalities are strict

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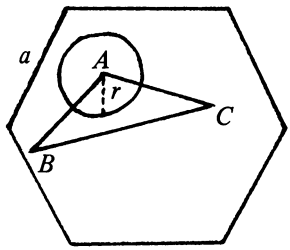

FIGURE 1. A regular hexagon containing the union of a triangle and a circle (except for the degenerate case $i = j = k - 1$), then the path is "the" minimal spanning tree.

LEMMA 4. *Suppose that $B_1B_2\cdots B_n$, $n \geq 3$, is a Steiner path and $B_1B_2, \ldots, B_nB_1$ form a convex polygon. Then $n \leq 7$ and $\sphericalangle B_1 + \sphericalangle B_n = 60n - 120$.*

PROOF.

$$\sphericalangle B_1 + \sphericalangle B_n = (n-2)180 - \sum_{i=2}^{n-1} \sphericalangle B_i = (n-2)180 - (n-2)120 = 60n - 120.$$

Now $n \leq 7$ follows from the assumption of convex polygon.

The following lemma will be critically needed in proving the main results.

LEMMA 5. *Let H denote a regular hexagon of side a (the length). Let R denote the region which consists of an isosceles triangle $\triangle ABC$ with $d(A,B) = d(A,C)$, and a circle with center A and radius r (see Figure 1). Suppose that R is contained in H and $d(B,C) \geq a$. Then*

$$d(A,B) \leq \sqrt{\left(\sqrt{3}a - r\right)^2 + (a/2)^2} \equiv z.$$

PROOF. Let $h_1, \ldots, h_6$ denote the six corners of H. Let H' be a smaller regular hexagon with the six corners $h'_1, \ldots, h'_6$ such that H' and H are concentric and the corresponding sides are parallel with distance r (see Figure 2).

Then Lemma 5 can be restated as follows:

"*Show that for any isosceles triangle $\triangle ABC$, $d(A,B) = d(A,C)$, satisfying the conditions that* (i) *A is in H',* (ii) *B and C are in H,* (iii) *$d(B,C) \geq a$, we have $d(A,B) \leq z$ $(= d(g', h_3)$ where g' is the midpoint of $h'_5h'_6)$.*"

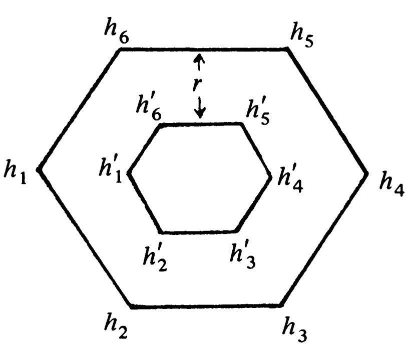

FIGURE 2. Two concentric regular hexagons

We may assume without loss of generality that A lies on the boundary of H' and at least one of B and C lies on the boundary of H. By symmetry we may further assume that A lies on $g'h_5'$. Let g be the midpoint of h_2h_3 and g'' the midpoint of h_1h_2. It is easy to see that the distances from g' to h_i, $i = 1, \ldots 6$, and the distances from h_5' to $g, h_3, h_4, h_5, h_6, h_7$ and g'' are all at most z. Thus by Lemma 2, the distance from g' to any point in H and the distance from h_5' to any point in the heptahedron $F = g''gh_3h_4h_5h_6h_1g''$ is at most z. Applying Lemma 2 once more, we see that the distance from A to any point in F is at most z. Thus the only points B and C which can violate the lemma must fall in the triangle gh_2g''. The diameter of this triangle, however, is at most $d(g, g'') < a$, and so the lemma is proven.

3. The main results. Let $A = A_1, \ldots, A_n$ be a set of points on a unit circle with center O. Define

$$m = \min\{[\alpha\beta + \sqrt{\alpha^2 + (1-\beta^2)/4}]/(\alpha^2 + 1/4), \gamma\}$$

where $\alpha = \sqrt{3} + 1 - 1/(2\rho)$, $\beta = 1 - (1-\rho)\pi/\rho$, $\gamma = 2(\sqrt{3}+1)/[(\sqrt{3}+1)^2 + 1/4] = .70832\ldots$ and ρ is the Steiner ratio.

THEOREM. *If the polygon $P_n = A_1A_2 \cdots A_n$ has at most one side longer than m, then the Steiner minimal tree for A is the sides of P_n excluding the longest one.*

PROOF. By Lemma 3 the minimal spanning tree for A is the sides of P_n excluding the longest one. We now prove that a Steiner minimal tree for A cannot have a Steiner point.

Suppose to the contrary that T is a Steiner minimal tree for A and T contains a Steiner point. It is well known [**4**] that the edges of a Steiner minimal tree can be uniquely decomposed at points of A into nonempty components in which every A_i is of degree 1 and is connected to a Steiner point if there is one in the component. Decompose T into such components $T_1, T_2, \ldots$. Then there exists a T_i containing a Steiner point. Now T_i partitions P_n into convex regions $Q_1, \ldots, Q_q$. Choose that region Q_w containing the center O of the circle. The boundary P_w of Q_w is made up of a subpath $A_i, A_{i+1}, \ldots, A_j$ of the polygon P_n and a Steiner path $A_i = s_0, s_1, \ldots, s_k, s_{k+1} = A_j$, where $s_1, \ldots, s_k$ are Steiner points of T. Define

$$u_w = d(s_c, s_{c+1}) = \max_{l=0,\ldots,k} d(s_l, s_{l+1}),$$
$$v_w = d(A_t, A_{t+1}) = \max_{l=i,\ldots,j-1} d(A_l, A_{l+1}).$$

Then $v_w \geq u_w$, since otherwise we could substitute the appropriate edge of the path $A_i, \ldots, A_j$ for the edge s_cs_{c+1} and shorten the length of T. Also, we have s_cs_{c+1} appearing in at least one other region $Q_{w'}$, where again $d(s_c, s_{c+1}) \leq u_{w'} \leq v_{w'}$. But since at most one of v_w and $v_{w'}$ is greater than m, then

$$u_w \leq \min\{v_w, v_{w'}\} \leq m.$$

We now construct a regular hexagon H with vertices, $h_1, \ldots, h_6$ and side u_w such that s_m, $m = \lceil k/2 \rceil$ ($\lceil x \rceil$ denotes the smallest integer not less than x), is a vertex of H and the two edges $[s_{m-1}, s_m]$ and $[s_m, s_{m+1}]$ overlap the sides of H. Then P_w is contained in H (see Figure 3).

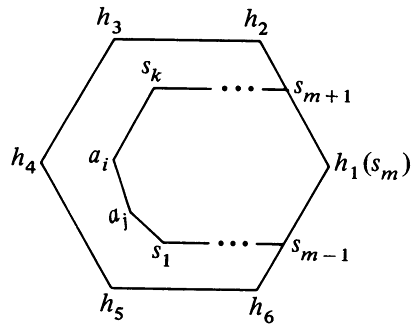

FIGURE 3. A regular hexagon containing P_w

We first consider the case that the center O is not in the polygon $P'_w = A_j s_1 \cdots s_k A_i$. Then P'_w is completely contained in the region enclosed by $A_1 A_j$ and $A_i A_j$, where $A_i A_j$ denote the shorter of the two arcs from A_i to A_j. If $d(A_i, A_j) < \sqrt{3}$, then $A_i A_j < 120°$. But this is impossible since if $k = 1$, then $\sphericalangle A_i + \sphericalangle A_j < 60$, and if $k > 1$, then $\sphericalangle A_i + \sphericalangle A_j < 60° + 60° = 120°$, both contradicting Lemma 4.

If $d(A_i, A_j) \geq \sqrt{3}$, then

$$2m \geq 2u_w \geq d(A_i, A_j) \geq \sqrt{3} \quad \text{by Lemma 1}$$

implies that $m \geq \sqrt{3}/2 > .708$, a contradiction to $\gamma < 1$.

Next we consider the case that center O lies in the polygon P'_w. Let D be the point on the Steiner path $A_j s_1 \cdots s_k A_i$ closest to O and let $d(D, O) = r$. Note that the circle with center O and radius r, as well as the triangle $\triangle O A_i A_j$, is contained in H. Furthermore, $d(A_i, A_j) \geq d(A_t, A_{t+1}) = v_w \geq u_w$. Hence the conditions of Lemma 5 are satisfied and we conclude

$$1 = d(O, A_i) \leq \sqrt{(\sqrt{3}u_w - r)^2 + (u_w/2)^2} \leq \sqrt{(\sqrt{3}m - r)^2 + (m/2)^2},$$

or equivalently,

$$r \leq \sqrt{3}m - \sqrt{1 - (m/2)^2}.$$

If $1 - r < m$, then we have

$$1 - m < r \leq \sqrt{3}m - \sqrt{1 - (m/2)^2}$$

or

$$m > 2(\sqrt{3} + 1)/[(\sqrt{3} + 1)^2 + 1/4] = \gamma,$$

a contradiction to the definition of m.

If $1 - r \geq m$, partition T into T' and T'' at D. Let A be partitioned into A' and A'' accordingly. Then T' and T'' are Steiner minimal trees for the point-sets $A' \cup \{D\}$ and $A'' \cup \{D\}$, respectively. Let $L(t)$ denote the length of the tree t. Let p denote the perimeter of P_n and let a and b denote the longest and the second longest side of P_n. Note that $b \leq m \leq 1 - r$. Furthermore, consider $\triangle A_i O D$ for each $A_i \in A$. Then we have $d(A_i, D) \geq d(O, A_i) - d(O, D) = 1 - r \geq b$. It follows that

$$L_M(A' \cup \{D\}) + L_M(A'' \cup \{D\}) \geq p - a - b + 2(1 - r).$$

Therefore,

$$\begin{aligned} p - a \geq L(T) &= L(T') + L(T'') \\ &\geq \rho L_M(A' \cup \{D\}) + \rho L_M(A'' \cup \{D\}) \geq \rho[p - a - b + 2(1 - r)], \end{aligned}$$

or

$$\begin{aligned} 2\rho(1 - r) &\leq (1 - \rho)(p - a) + \rho b \leq (1 - \rho)(p - b) + \rho b \\ &\leq (1 - \rho)p + (2\rho - 1)b < (1 - \rho)2\pi + (2\rho - 1)m. \end{aligned}$$

Hence,

$$r > 1 - (1 - \rho)\pi/\rho - (2\rho - 1)m/(2\rho).$$

But we have shown that $r \leq \sqrt{3}m - \sqrt{1 - (m/2)^2}$. Combining these two inequalities, we obtain

$$\alpha m - \beta > \sqrt{1 - (m/2)^2}$$

where α and β are defined at the beginning of this section (note that $\alpha > 1/2$ and $\beta < 1$). Solving for m, we obtain

$$m > [\alpha\beta + \sqrt{\alpha^2 + (1 - \beta^2)/4}]/(\alpha^2 + 1/4),$$

or

$$m < [\alpha\beta - \sqrt{\alpha^2 + (1 - \beta^2)/4}]/(\alpha^2 + 1/4) < 0,$$

both lead to a contradiction to the definition of m. We have thus proved that T cannot have a Steiner point. □

Substituting ρ by .8 in the Theorem, we obtain

COROLLARY. *The Steiner minimal tree for A is the sides of P_n excluding the longest one if there is at most one side longer than* .55762....

4. Some concluding remarks. Graham [**5**] conjectured that if A is a set of points on a circle such that the largest angle formed by two consecutive points at the center is at most $\pi/3$, then no Steiner minimal tree for A contains a Steiner point. The updated version of the conjecture is that at most one such angle can be larger than $\pi/3$. If the conjecture is stated using side rather than angle, then the condition becomes that at most one side can be longer than the radius. In a private communication Graham informed us that he can use the "probe" idea given in [**3**] to prove this conjecture if the degree $\pi/3$ is replaced by $20°$, which corresponds to a side of length .35 assuming radius one. Thus the result of this paper can be used to improve on the length .35 significantly. On the other hand, we note that m is at most .70832... even if one can prove the conjecture of Gilbert and Pollak that $\rho = \sqrt{3}/2$. Therefore, the approach in this paper is not strong enough to prove the conjecture of Graham.

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ACADEMIA SINICA, BEIJING, CHINA

AT&T BELL LABORATORIES, MURRAY HILL, NEW JERSEY 07974

NORTH EAST HEAVY MACHINERY INSTITUTE, QIQIHAER, CHINA