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STRUCTURE OF BANACH ALGEBRAS A SATISFYING $Ax^2 = Ax$ FOR EVERY $x \in A$

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ABSTRACT. We give a complete characterization of Banach algebras which satisfy the condition $Ax^2 = Ax$ for every $x \in A$.

I. Introduction. C. Lepage [5] showed that a unital Banach algebra A, such that $Ax = Ax^2$ for every $x \in A$, is semisimple and commutative. J. Duncan and A. W. Tullo [4] proved that such an algebra is in fact finite dimensional (hence isomorphic to C^n for some $n \geqslant 0$). B. Aupetit obtained in [1] a theorem (Theorem 1, p. 58) which extends the results of Lepage and Duncan-Tullo, but did not study the nonunital case. In their survey paper [2] V. A. Belfi and R. S. Doran asked to what extent the conclusion remains true for nonunital algebras. The latter author showed [6] that the result of Lepage remains valid for Banach algebras with bounded approximate identities.

We completely describe here the structure of Banach algebras A satisfying $Ax^2 = Ax$ for every $x \in A$. If A is semisimple, then $A \simeq C^n$ for some $n \geqslant 0$. In general A is isomorphic to $C^n \oplus R$ (Theorem 3.2) for some $n \geqslant 0$, where R is the radical of A and where $AR = \{0\}$ (i.e., $ax = 0$ for every $a \in A$ and every $x \in R$). These conditions are necessary and sufficient. These results, which conclude a work initiated by the second author in [6], were obtained in June 1983 at the University of Montreal. The first author wishes to thank the Department of Mathematics of the University of Montreal for their kind hospitality, and the authors wish to thank the referee for his careful checking of the original manuscript.

II. The commutative semisimple case. Throughout this paper we set $A_e = A \oplus Ce$ if A is a nonunital Banach algebra, and $A_e = A$ if A is a unital Banach algebra. The spectrum $\mathrm{Sp}_A(x)$ of an element x of A (denoted by $\mathrm{Sp}(x)$ if no confusion is possible) is by definition the spectrum of x in A_e, so that $0 \in \mathrm{Sp}_A(x)$ if A is not unital.

PROPOSITION 2.1. *Let A be a commutative Banach algebra. If A possesses infinitely many characters, then there exists $x \in A$ such that 0 is an accumulation point of* $\mathrm{Sp}(x)$.

PROOF. It follows from a well-known result of Kaplansky that $\mathrm{Sp}(x)$ is infinite for some $x \in A$. To see this, choose for example a sequence X_n of distinct characters of A and put $\Omega_{n,m} = \{x \in A \mid X_n(x) \neq X_m(x)\}$ for $n \neq m$. Then $\Omega_{n,m}$ is dense and

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open in A, so $\Omega = \bigcap_{n \neq m} \Omega_{n,m}$ is dense in A and all elements of Ω have an infinite spectrum. Then there exists $\lambda \in C$, $x \in A$ and a sequence $\{\lambda_n\}$ of elements of $\mathrm{Sp}(x)$ such that $\lambda_n \to \lambda$ as $n \to \infty$, $\lambda_n \neq \lambda$ for each n.

If A possesses a unit element e, then 0 is an accumulation point of $\mathrm{Sp}(x - \lambda e)$. If A has no unit element, let X_0 be the character of A_e such that $A = \operatorname{Ker} X_0$.

If X_0 is an isolated point of the carrier space of A_e (this may happen for nonunital and nonsemisimple Banach algebras with compact carrier space), it follows from Shilov's idempotent theorem [**3**, p. 109, Theorem 5] that there exists an idempotent p of A_e such that $X(p) = 1$ if X is a character on A_e distinct from X_0 and such that $X_0(p) = 0$. Then $p \in A$ and $X(p) = 1$ for any character X on A. If x is as above, then $x - \lambda p$ has the desired property. Now assume that A is nonunital and that X_0 is not an isolated point of the carrier space of A_e. Put $\Omega_n = \{y \in A \mid 0 < |\lambda| < 1/n$ for some $\lambda \in \mathrm{Sp}(y)\}$ for $n \in N$. Then Ω_n is clearly open for each n. Take $z \in A$. Then $X_0(z) = 0$ and since X_0 is not isolated in the carrier space of A_e, there exists a character $X \neq X_0$ on A_e such that $|X(z)| < 1/n$. If $X(z) \neq 0$, then $z \in \Omega_n$. If $X(z) = 0$, there exists $u \in A$ such that $X(u) \neq 0$, and $z + \varepsilon u \in \Omega_n$ if ε is sufficiently small. So Ω_n is dense in A. Again using the category theorem, we see that $\bigcap_{n \in N} \Omega_n \neq \varnothing$ and elements of this set have the desired property.

PROPOSITION 2.2. *Let A be a commutative Banach algebra and let x be an element of A whose spectrum contains* 0.

(i) *If* 0 *is not isolated in* $\mathrm{Sp}(x)$, *then* $Ax^2 \neq Ax$.

(ii) *If* 0 *is isolated in* $\mathrm{Sp}(x)$ *and if A is semisimple*, *then* $Ax^2 = Ax$.

PROOF. (i) If 0 is not isolated in $\mathrm{Sp}(x)$, there exists a sequence $\{X_n\}$ of characters of A such that $X_n(x) \to 0$ as $n \to \infty$, and such that $X_n(x) \neq 0$ for each n.

If $Ax^2 = Ax$ there would exist $u \in A$ and $v \in A$ such that $x^2 = ux^2$ and $ux = vx^2$. Then $X_n(u) = 1$, and $|X_n(u)| = |X_n(x)| \cdot |X_n(v)| \leqslant |X_n(x)| \cdot \|v\|$ for each n. As $X_n(x) \to 0$, when $n \to \infty$, this is impossible.

(ii) Now assume that A is semisimple. If $\mathrm{Sp}(x) = \{0\}$, then $x = 0$, so that $Ax^2 = Ax$. If $\mathrm{Sp}(x)$ contains at least two points, and if 0 is an isolated point of $\mathrm{Sp}(x)$, then it follows from an easy version of Shilov's idempotent theorem [**3**, p. 36, Proposition 9] that there exists an idempotent p of A_e such that $X(p) = 0$ for every character X of A_e vanishing at x and such that $X(p) = 1$ for every character X of A_e which does not vanish at x.

If A is not unital, then $A = \operatorname{Ker} X_0$ for some character X_0 of A_e, and $X_0(p) = X_0(x) = 0$. So $p \in A$, even if A is not unital.

Since A is semisimple, we have $px = x$. Set $D = pA$. Then D is a commutative unital Banach algebra. Let ϕ be a character on D. The map X: $a \to \phi(ap)$ is a character on A_e since $p \in pA$ and $X|D = \phi$. So $X(p) = \phi(p) = 1$, and $\phi(x) = X(x) \neq 0$. We thus see that x possesses an inverse x^{-1} in D, and $Ax = Apx = Ax^{-1} \cdot x^2 \subset Ax^2$. So $Ax = Ax^2$, and the proposition is proved.

COROLLARY 2.3. *Let A be a commutative semisimple Banach algebra. If $Ax^2 = Ax$ for every $x \in A$, then A is isomorphic to C^n for some $n \geqslant 0$.*

PROOF. It follows from Propositions 2.1 and 2.2 that A possesses only finitely many characters. So the carrier space of A is compact (we exclude the obvious case where $A = \{0\}$) and A is unital, since it is semisimple [**3**, p. 109, Corollary 6]. Denote by $X_1, \ldots, X_n$ the characters of A. There exists a family $(e_i)_{i \leqslant n}$ of idempotents of A such that $X_i(e_i) = 1$, $X_i(e_j) = 0$ for $i \neq j$ and the map $(\lambda_1, \ldots, \lambda_n) \to \sum_{i=1}^{n} \lambda_i e_i$ is an isomorphism from C^n onto A (we could, in fact, use the Wedderburn structure theorem [**3**, p. 134]).

REMARK 2.4. Assertion (ii) of Proposition 2.2 might fail if A is not semisimple. Consider a trivial Banach algebra R (where $yz = 0$ for every $y, z \in R$). Put $B = R \oplus Cf$, where $f^2 = f$, $Rf = fR = 0$ and put $A = B_e = R \oplus Cf \oplus Ce$. Now put $u = f + y$, where $y \in R$, $y \neq 0$. Then $u \in Au$, $u^2 = f$. If $\lambda, \mu \in C$, $z \in R$, we have $(\lambda e + \mu f + z)f^2 = (\lambda + \mu)f \neq f + y$, so that $Au \neq Au^2$, and $\mathrm{Sp}(u) = \{0, 1\}$ so that 0 is an isolated point of $\mathrm{Sp}(u)$.

III. The general case.

LEMMA 3.1. *Let A be a Banach algebra such that $Ax^2 = Ax$ for every $x \in A$ and let R be the radical of A. Then*:

(i) $ax = 0$ *for each $x \in R$ and each $a \in A$.*

(ii) *The quotient algebra A/R is commutative.*

PROOF. Let y be any element of R. We can find $u, v \in A$ such that $y^2 = uy^2 = vy^3$. So $y^2(e - vy) = 0$. Since $y \in R$, $e - vy$ is invertible in A and $y^2 = 0$. Then $Ay = Ay^2 = \{0\}$.

Now let π: $A \to \mathscr{L}(\mathscr{M})$ be any irreducible representation of A on an A-module $\mathscr{M}$. If $\dim \mathscr{M} \geqslant 2$, pick two linearly independent elements e_1, e_2 of $\mathscr{M}$.

It follows from [**3**, p. 128, Corollary 5] that there exists $u \in A$ such that $\pi(u)e_1 = e_2$, $\pi(u)e_2 = 0$. But there exists $v \in A$ such that $\pi(v)e_2 = e_2$. Now for every $w \in A$ we have

$$\pi(wu^2)e_1 = \pi(w)\pi(u)e_2 = 0 \neq e_2 = \pi(v)e_2 = \pi(vu)e_1.$$

So $vu \notin Au^2$, and we see that if $Ax^2 = Ax$ for each $x \in A$ then all irreducible representations of A have dimension 1. Then $\pi(A)$ is isomorphic to C, so that $\pi(x)\pi(y) = \pi(y)\pi(x)$ and $xy - yx \in \operatorname{Ker} \pi$ for all irreducible representations of A and for every $x, y \in A$. So $xy - yx \in R$ [**3**, p. 124, Proposition 14].

We now obtain the following theorem.

THEOREM 3.2. *Let A be a Banach algebra. Then $Ax = Ax^2$ for every $x \in A$ if and only if $A = B \oplus R$, where $AR = \{0\}$ and where B is isomorphic to C^n for some $n \geqslant 0$.*

PROOF. Denote by R the radical of A. Then $D = A/R$ is commutative, semisimple, and satisfies $Dx = Dx^2$ for every $x \in D$. So $D \simeq C^n$ for some $n \geqslant 0$. We omit the trivial case $D = \{0\}$ and we can find a family $f_1, \ldots, f_n$ of idempotents of D such that $f_1 \cdot f_j = 0$ if $i \neq j$ and such that $D = Cf_1 \oplus \cdots \oplus Cf_n$. Denote by π: $A \to A/R$ the natural surjection and choose $g_1, \ldots, g_n \in A$ such that $\pi(g_i) = f_i$, $1 \leqslant i \leqslant n$. Then $\pi(g_i^2 - g_i) = 0$, $g_i^2 - g_i \in R$ and $g_i^2 - g_i^3 = g_i(g_i - g_i^2) = 0$ by the lemma. So $g_i^2 = g_i^3 = g_i^4$. Put $e_i = g_i^2$. Then $\pi(e_i) = \pi(g_i) + \pi(g_i^2 - g_i) = \pi(g_i) = f_i$ for each $i \leqslant n$. Also, $\pi(e_i e_j) = f_i f_j = 0$, so that $e_i e_j \in R$ and $e_i e_j = e_i(e_i e_j) = 0$ for $i \neq j$. Denote by B the linear span of $\{e_1, \ldots, e_n\}$. Then $B \simeq C^n$, $B \cap R = \{0\}$. If $x \in A$ we can write $\pi(x) = \lambda_1 f_1 + \cdots + \lambda_n f_n$, so that $x - \lambda_1 e_1 - \cdots - \lambda_n e_n \in R$. So $A = B \oplus R$ and the condition of the theorem is necessary.

Conversely if $A = B \oplus R$, where B and R satisfy the above condition, take a, $b \in B$ and x, $y \in R$. There exists $d \in B$ such that $a = da^2$. We have

$$\begin{aligned}(bd + yd)(a + x)^2 &= (bd + yd)(a^2 + xa) = bda^2 + yda^2 \\ &= ba + ya = (b + y)(a + x).\end{aligned}$$

So $Au = Au^2$ for every $u \in A$, and the proof is complete.

COROLLARY 3.3. *Let A be a semisimple Banach algebra. Then $Ax = Ax^2$ for every $x \in A$ if and only if A is isomorphic to C^n for some $n \geqslant 0$.*

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