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CYCLIC VECTORS FOR BACKWARD HYPONORMAL WEIGHTED SHIFTS

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ABSTRACT. A unilateral weighted shift T on a Hilbert space H is an operator such that $Te_n = w_n e_{n+1}$ for some orthonormal basis $\{e_n\}_{n=0}^{\infty}$ and weight sequence $\{w_n\}_{n=0}^{\infty}$. If we assume $w_n > 0$, for all n, and let $\beta(n) = w_0 \cdots w_{n-1}$ for $n > 0$ and $\beta(0) = 1$, then T is unitarily equivalent to $f \mapsto zf$ on the weighted space $H^2(\beta)$ of formal power series $\sum_{n=0}^{\infty} \hat{f}(n) z^n$ such that $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 [\beta(n)]^2 < \infty$. Regarding T as multiplication for z on the space $H^2(\beta)$, it is shown that, if $w_n \uparrow 1$ and f is analytic in a neighborhood of the unit disk, then either f is cyclic for T^* or f is contained in a finite-dimensional T^*-invariant subspace. This was shown—by different methods—for the unweighted shift operator by Douglas, Shields, and Shapiro [**2**]. It is also shown that every finite-dimensional T^*-invariant subspace is of the form

$$\left((z-\alpha_1)^{n_1} \cdots (z-\alpha_k)^{n_k} H^2(\beta)\right)^{\perp},$$

for some $\alpha_1, \ldots, \alpha_k$ in the unit disk and $n_1, \ldots, n_k$ positive integer.

1. Introduction and notation. Let U be the unilateral shift on the Hardy space H^2. It follows from Beurling's theorem that a function f is noncyclic for U^* if and only if $f \in (\varphi H^2)^{\perp}$ for some inner function φ. But this is not a very useful condition for determining whether a given function is cyclic for U^*. In [**2**], Douglas, Shields and Shapiro give a much more useful characterization (Theorem 2.2.1), which has as one of its consequences Theorem 2.2.4, which states that if f is analytic in a neighborhood of the unit disk then f is either cyclic or a rational function. Since the functions which are contained in a finite-dimensional U^*-invariant subspace are precisely the rational functions, Theorem 2.2.4 can be restated as follows.

THEOREM 0. *If f is analytic in a neighborhood of the unit disk, then either f is cyclic for U^* or f is contained in a finite-dimensional U^*-invariant subspace.*

The main result of this paper is that this is true for any hyponormal weighted shift with unit norm. This is proved in §2; in §3 the finite-dimensional invariant subspaces for hyponormal weighted shifts are characterized.

Throughout this paper T will be a hyponormal weighted shift with unit norm. We use the weighted space notation used in [**3**], and we assume that T is the operator on $H^2(\beta)$ defined by $Tf = zf$. If S is an operator, $\sigma(S)$, $\sigma_{\mathrm{p}}(S)$, and $\sigma_{\mathrm{ap}}(S)$ will denote

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its spectrum, point spectrum, and approximate spectrum, respectively. If $f \in H^2(\beta)$, then $[f]_*$ will be the smallest T^*-invariant subspace containing f.

Since T is hyponormal with unit norm, we have $w_n \uparrow 1$, so $[\beta(n)]^{1/n} \to 1$. It follows that any function in $H^2(\beta)$ is analytic on the unit disk. If $|\alpha| < 1$ and n is a nonnegative integer, let

$$K_{\alpha, n} = \sum_{j=n}^{\infty} \frac{j \cdots (j-n+1)}{[\beta(j)]^2} \bar{\alpha}^{j-n} z^j.$$

Then $K_{\alpha, n} \in H^2(\beta)$ and, for any f in $H^2(\beta)$, we have

$$\langle f, K_{\alpha, n} \rangle = \sum_{j=n}^{\infty} j \cdots (j-n+1) \hat{f}(j) \alpha^{j-n} = f^{(n)}(\alpha).$$

Since the function $K_{\alpha,0}$ will be used particularly often, when it is convenient we will call it K_α.

2. The main result.

THEOREM 1. *If T is a hyponormal unilateral weighted shift with unit norm and f is analytic in a neighborhood of the unit disk, then either f is cyclic or f is contained in a finite-dimensional T^*-invariant subspace.*

We prove this by way of the following lemmas.

LEMMA 1. *If $|\alpha| < 1$ and $f \in H^2(\beta)$, then*
(i) $(T^* - \bar{\alpha}) f = 0$ *if and only if f is a constant multiple of K_α;*
(ii)

$$(T^* - \bar{\alpha}) \left(\frac{1}{n!} K_{\alpha, n} \right) = \frac{1}{(n-1)!} K_{\alpha, n-1} \quad \textit{for any } n > 0.$$

PROOF. (i) For any g in $H^2(\beta)$, we have $\langle g, (T^* - \bar{\alpha}) K_\alpha \rangle = \langle (z-\alpha) g, K_\alpha \rangle = 0$, so $(T^* - \bar{\alpha}) K_\alpha = 0$. Conversely, suppose $(T^* - \bar{\alpha}) f = 0$. Since the polynomials are dense in $H^2(\beta)$ and $(T - \alpha)$ is bounded below (by Proposition 8.13 in [**1**]), if g is in $H^2(\beta)$, then the function $(g - g(\alpha))/(z - \alpha)$ is also in $H^2(\beta)$. Thus if $g \in H^2(\beta)$, then

$$\begin{aligned} \langle g, f \rangle &= \left\langle \frac{g - g(\alpha)}{z - \alpha} (z - \alpha), f \right\rangle + g(\alpha) H^2(\alpha) \langle 1, f \rangle \\ &= \left\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \bar{\alpha}) f \right\rangle + g(\alpha) \langle 1, f \rangle = g(\alpha) \langle 1, f \rangle. \end{aligned}$$

Therefore $f = \langle 1, f \rangle K_\alpha$.

(ii) If $g \in H^2(\beta)$, then

$$\begin{aligned} \left\langle g, (T^* - \bar{\alpha}) \left(\frac{1}{n!} K_{\alpha, n} \right) \right\rangle &= \left\langle (z - \alpha) g, \frac{1}{n!} K_{\alpha, n} \right\rangle = \frac{1}{n!} ((z - \alpha) g)^{(n)}(\alpha) \\ &= \frac{1}{(n-1)!} g^{(n-1)}(\alpha) = \left\langle g, \frac{1}{(n-1)!} K_{\alpha, n-1} \right\rangle. \qquad \square \end{aligned}$$

LEMMA 2. *If M is an invariant subspace for T^* then*

$$\sigma_{\mathrm{ap}}(T^*|M) \cap \{|z|<1\} = \sigma_{\mathrm{p}}(T^*|M).$$

PROOF. Let $\bar{\alpha} \in \sigma_{\mathrm{ap}}(T^*|M)$ with $|\alpha| < 1$. Then there exists a sequence of functions $\{f_n\}$ in M such that $\|f_n\| = 1$ and $\|(T^* - \bar{\alpha})f_n\| \to 0$. Let $f_n = c_n K_\alpha + g_n$, where $g_n \perp K_\alpha$. Then $\|(T^* - \bar{\alpha})g_n\| \to 0$. The subspace $(T^* - \bar{\alpha})\{K_\alpha\}^\perp$ is the range of $T^* - \bar{\alpha}$, which is closed by Proposition 8.13 of [1], so $T^* - \bar{\alpha}$: $\{K_\alpha\}^\perp \to (T^* - \bar{\alpha})\{K_\alpha\}^\perp$ is invertible. This implies $T^* - \bar{\alpha}$ is bounded below on $\{K_\alpha\}^\perp$, so $\|g_n\| \to 0$. Since $\|f_n\|$ and $\|g_n\|$ are bounded, the sequence $\{c_n\}$ is bounded, so it has a convergent subsequence $\{c_{n_k}\}$. Let $c = \lim_{k\to\infty} c_{n_k}$. Then $f_{n_k} \to cK_\alpha$, so $\bar{\alpha} \in \sigma_{\mathrm{p}}(T^*|M)$. □

LEMMA 3. *If $f \in H^2(\beta)$ and there is a constant C such that $\|q(T^*)f\| \leqslant C\|q\|$ for all polynomials q, then*

$$\sigma(T^*|[f]_*) \cap \{|z|<1\} = \sigma_{\mathrm{p}}(T^*|[f]_*).$$

PROOF. Let $\bar{\alpha} \in \sigma(T^*|[f]_*)$ with $|\alpha| < 1$. By Lemma 2, if $\bar{\alpha} \in \sigma_{\mathrm{ap}}(T^*|[f]_*)$, then $\bar{\alpha} \in \sigma_{\mathrm{p}}(T^*|[f]_*)$, so assume $\bar{\alpha}$ is in the compression spectrum. Then there exists a nonzero function g in $[f]_* \ominus (T^* - \bar{\alpha})[f]_*$. Let $g^*(z) = \overline{g(\bar{z})}$, and let $\{q_n\}$ be a sequence of polynomials such that $q_n \to g^*$ (in $H^2(\beta)$). Then

$$\|q_n(T^*)f - q_m(T^*)f\| \leqslant C\|q_n - q_m\| \to 0$$

as $n, m \to \infty$, so $\{q_n(T^*)f\}$ converges. Let $h = \lim_{n\to\infty} q_n(T^*)f$. Then $h \in [f]_*$ and, for any nonnegative integer k, we have

$$\begin{aligned}\langle (T^* - \alpha)h, z^k\rangle &= \langle h, (z-\alpha)z^k\rangle = \lim_{n\to\infty} \langle q_n(T^*)f, (z-\alpha)z^k\rangle \\ &= \lim_{n\to\infty} \langle f, (z-\alpha)z^k\overline{q_n(\bar{z})}\rangle = \langle f, (z-\alpha)z^k g\rangle \\ &= \langle (T^* - \bar{\alpha})T^{*k}f, g\rangle = 0,\end{aligned}$$

so $(T^* - \bar{\alpha})h = 0$.

If $h = 0$, then, for any nonnegative integer k, we have

$$\begin{aligned}0 = \langle h, z^k\rangle &= \lim_{n\to\infty} \langle q_n(T^*)f, z^k\rangle \\ &= \lim_{n\to\infty} \langle f, q_n(\bar{z})z^k\rangle = \langle f, gz^k\rangle = \langle T^{*k}f, g\rangle,\end{aligned}$$

so $g \perp [f]_*$, contradicting the assumption that g is a nonzero function in $[f]_*$. Thus $h \neq 0$, so $\bar{\alpha} \in \alpha_{\mathrm{p}}(T^*|[f]_*)$.

COROLLARY 1. *If T is subnormal and $f \in H^\infty(\beta)$, then*

$$\sigma(T^*|[f]_*) \cap \{|z|<1\} = \alpha_{\mathrm{p}}(T^*|[f]_*).$$

PROOF. Let M_f be the operator on $H^2(\beta)$ defined by $M_f g = fg$. Then M_f is bounded. Let q be a polynomial and let $q^*(z) = \overline{q(\bar{z})}$. Then since T is subnormal, $q^*(T)$ is also subnormal, so, in particular, it is hyponormal, so

$$\begin{aligned}\|q(T^*)f\| &= \|(q^*(T))^*f\| \leqslant \|q^*(T)f\| \\ &= \|q^*(z)f\| \leqslant \|M_f\|\|q^*\| = \|M_f\|\|q\|.\end{aligned}$$

LEMMA 4. *Let f be a nonzero function in $H^2(\beta)$, such that*

$$R^{2n}\sum_{k=0}^{\infty}\frac{[\beta(k+n)]^4}{[\beta(k)]^2}|\hat{f}(k+n)|^2 \to 0$$

as $n\to\infty$, for some $R>1$. If $1/R<r<1$, then the intersection $\sigma(T^*|[f]_*)\cap\{|z|<r\}$ is nonempty.

PROOF. Suppose $\sigma(T^*|[f]_*)\cap\{|z|<r\}=\varnothing$. Then $(T^*|[f]_*)^{-1}$ exists and $\sigma((T^*|[f]_*)^{-1})\subseteq\{|z|<1/r\}$. Hence, $\lim_{n\to\infty}\|(T^*|[f]_*)^{-n}\|^{1/n}\leqslant 1/r$ so, since $1/r<R$, there exists N such that $\|(T^*|[f]_*)^{-n}\|^{1/n}\leqslant R$, for all $n\geqslant N$. Thus, for $n\geqslant N$, we have $\|(T^*|[f]_*)^{-n}\|\leqslant R^n$. In particular,

$$\|f\|=\|(T^*|[f]_*)^{-n}T^{*n}f\|\leqslant R^n\|T^{*n}f\|,$$

so

$$\|f\|^2\leqslant R^{2n}\|T^{*n}f\|^2=R^{2n}\sum_{k=0}^{\infty}\frac{[\beta(k+n)]^4}{[\beta(k)]^2}|\hat{f}(k+n)|^2\to 0$$

as $n\to\infty$, contradicting the assumption that f is nonzero. □

If $f\in H^2(\beta)$ and $R>0$, let f_R be the function defined by $f_R(z)=f(Rz)$.

LEMMA 5. *Let $f\in H^2(\beta)$ and $R>1$. If $f_R\in H^2(\beta)$ and $|\alpha|<1$, then*

$$((T^*-\bar{\alpha})f)_R\in H^2(\beta).$$

PROOF. Let $h=(T^*-\bar{\alpha})f$. Then

$$\hat{h}(n)=\hat{f}(n+1)\frac{\beta(n+1)}{\beta(n)}-\bar{\alpha}\hat{f}(n),$$

so

$$\sum_{n=0}^{\infty}|\hat{h}(n)|^2R^{2n}[\beta(n)]^2\leqslant 2\left(\sum_{n=0}^{\infty}[\beta(n+1)]^2|\hat{f}(n+1)|^2R^{2n}\right.$$
$$\left.+|\alpha|^2\sum_{n=0}^{\infty}[\beta(n)]^2|\hat{f}(n)|^2R^{2n}\right)<\infty,$$

so $h_R\in H^2(\beta)$. □

LEMMA 6. *If $|\alpha|<1$ and $K_{\alpha,n}\in[(T^*-\bar{\alpha})^mf]_*$, then $K_{\alpha,n+m}\in[f]_*$.*

PROOF. By induction it suffices to show that if $K_{\alpha,n}\in[(T^*-\bar{\alpha})f]_*$, then $K_{\alpha,n+1}\in[f]_*$. If $K_{\alpha,n}\in[(T^*-\bar{\alpha})f]_*$, then $K_{\alpha,n}\in[f]_*$, so since

$$(1/n!)(T^*-\bar{\alpha})^nK_{\alpha,n}=K_\alpha$$

(by Lemma 1), it follows that $K_\alpha\in[f]_*$. Let Q be the orthogonal projection from $[f]_*$ to $[f]_*\cap\{K_\alpha\}^\perp$. Since $K_{\alpha,n}\in[(T^*-\bar{\alpha})f]_*$, there exists a sequence of polynomials $\{q_k\}$ such that $q_k(T^*)(T^*-\bar{\alpha})f\to K_{\alpha,n}$ (as $k\to\infty$). Let $f_k=Qq_k(T^*)f$. Then $f_k\in[f]_*$ and $(T^*-\bar{\alpha})f_k\to K_{\alpha,n}$.

The sequence $\{\langle 1, f_k\rangle\}$ is bounded, since $|\langle 1, f_k\rangle| \leqslant \|f_k\|$ and $T^* - \bar{\alpha}$ is bounded below on $\{K_\alpha\}^\perp$. Hence, it has a convergent subsequence $\{\langle 1, f_{k_j}\rangle\}$. Let $d = \lim_{j\to\infty}\langle 1, f_{k_j}\rangle$. If $g \in H^2(\beta)$, then

$$\begin{aligned}\langle g, f_{k_j}\rangle &= \left\langle \frac{g - g(\alpha)}{z-\alpha}(z-\alpha), f_{k_j}\right\rangle + \langle g(\alpha), f_{k_j}\rangle \\ &= \left\langle \frac{g - g(\alpha)}{z-\alpha}, (T^* - \bar{\alpha})f_{k_j}\right\rangle + g(\alpha)\langle 1, f_{k_j}\rangle \\ &\to \left(\frac{g - g(\alpha)}{z-\alpha}\right)^{(n)}(\alpha) + dg(\alpha) \\ &= \frac{1}{n+1} g^{(n+1)}(\alpha) + dg(\alpha).\end{aligned}$$

Thus $f_{k_j} \to (1/(n+1))K_{\alpha, n+1} + dK_\alpha$ weakly as $j \to \infty$, so $K_{\alpha, n+1} \in [f]_*$. □

PROOF OF THEOREM 1. Let f be a function analytic in a neighborhood of the unit disk and not contained in a finite-dimensional T^*-invariant subspace. Then there exists $R > 1$ such that $f_R \in H^2(\beta)$.

Let q be a polynomial with $q(z) = \sum_{k=0}^N a_k z^k$. Then

$$\begin{aligned}\|q(T^*)f\|^2 &= \left\| \sum_{n=0}^{N} a_k \sum_{n=0}^{\infty} \frac{[\beta(k+n)]^2}{[\beta(n)]^2} \hat{f}(k+n) z^n \right\|^2 \\ &= \sum_{n=0}^{\infty} \left| \sum_{k=0}^{N} a_k \frac{[\beta(k+n)]^2}{[\beta(n)]^2} \hat{f}(k+n)\right|^2 [\beta(n)]^2 \\ &\leqslant \sum_{n=0}^{\infty} \left(\sum_{k=0}^{N} |a_k| \frac{[\beta(k+n)]^2}{[\beta(n)]^2} |\hat{f}(k+n)|\right)^2 [\beta(n)]^2 \\ &\leqslant \sum_{n=0}^{\infty} \left(\sum_{k=0}^{N} |a_k|^2 [\beta(k)]^2\right) \\ &\quad \times \left(\sum_{k=0}^{N} |\hat{f}(k+m)|^2 \frac{[\beta(k+n)]^4}{[\beta(k)]^2[\beta(n)]^4}\right)[\beta(n)]^2 \\ &= \|q\|^2 \sum_{n=0}^{\infty}\sum_{k=0}^{N} |\hat{f}(k+n)|^2 \frac{[\beta(k+n)]^4}{[\beta(k)]^2[\beta(n)]^2}.\end{aligned}$$

So to show that there exists a constant C such that $\|q(T^*)f\| \leqslant C\|q\|$ for any polynomial q, it is enough to show that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |\hat{f}(k+n)|^2 \frac{[\beta(k+n)]^4}{[\beta(k)]^2[\beta(n)]^2} < \infty.$$

Let $1 < R' < R$. Then there is a constant C_1 such that $1/\beta(n) \leqslant C_1(R')^n \leqslant C_1(R')^{n+k}$ for all nonnegative integers n and k. Then since $[\beta(k+n)]^2/[\beta(k)]^2 \leqslant 1$, we get

$$|\hat{f}(k+n)|^2 \frac{[\beta(k+n)]^4}{[\beta(k)]^2[\beta(n)]^2} \leqslant C_1^2 |\hat{f}(k+n)|^2 [\beta(k+n)]^2 (R')^{2(k+n)}.$$

Since $R' < R$, there is a constant C_2 such that $(n+1)(R')^{2n} \leqslant C_2 R^{2n}$, so

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |\hat{f}(k+n)|^2 [\beta(k+n)]^2 (R')^{2(k+n)} &= \sum_{n=0}^{\infty} (n+1) |\hat{f}(n)|^2 [\beta(n)]^2 (R')^{2n} \\ &\leqslant \sum_{n=0}^{\infty} |\hat{f}(n)|^2 [\beta(n)]^2 C_2 R^{2n} < \infty.\end{aligned}$$

Fix $1/R < r < 1$ and let $1/r < R_1 < R_2 < R$. Then for sufficiently large n, we have $|\hat{f}(n)| \leqslant 1/R_2^n$, so for such an n,

$$\begin{aligned} R_1^{2n} \sum_{k=0}^{\infty} \frac{[\beta(k+n)]^4}{[\beta(k)]^2} |\hat{f}(k+n)|^2 &\leqslant R_1^{2n} \sum_{k=0}^{\infty} \frac{[\beta(k+n)]^4}{[\beta(k)]^2} \left(\frac{1}{R_2^2}\right)^{2(k+n)} \\ &= \left(\frac{R_1}{R_2}\right)^{2n} \sum_{k=0}^{\infty} \frac{[\beta(k+n)]^4}{[\beta(k)]^2} \left(\frac{1}{R_2}\right)^{2k} \leqslant \left(\frac{R_1}{R_2}\right)^{2n} \sum_{k=0}^{\infty} \left(\frac{1}{R_2}\right)^{2k} \to 0 \end{aligned}$$

as $n \to \infty$, so by Lemmas 3 and 4, the intersection $\{|z| < r\} \cap \sigma_p(T^*|[f]_*)$ is nonempty.

Choose α_0 such that $\bar{\alpha}_0 \in \{|z| < r\} \cap \sigma_p(T^*|[f]_*)$. If α_k, $k = 0, \ldots, m-1$, are defined, let $f_m = (T^* - \bar{\alpha}_0) \cdots (T^* - \bar{\alpha}_{m-1}) f$. Since f is not contained in a finite-dimensional T^*-invariant subspace, it follows that $f_m \neq 0$. By Lemma 5 we have $(f_m)_R \in H^2(\beta)$ so, again by Lemmas 3 and 4, the intersection $\{|z| < r\} \cap \sigma_p(T^*|[f_m]_*)$ is nonempty, so choose α_m such that $\bar{\alpha}_m \in \{|z| < r\} \cap \sigma_p(T^*|[f_m]_*)$. In this way we obtain a sequence $\{\alpha_k\}$ of points in the disk $\{|z| < r\}$.

Suppose α occurs in $\{\alpha_k\}$ at least j times, and let N be the positive integer such that the jth occurrence of α in $\{\alpha_k\}$ is α_N. Then

$$K_\alpha \in [f_{N-1}]_* = \left[\prod_{k=0}^{N-1} (T^* - \bar{\alpha}_k) f\right]_* \subseteq \left[(T^* - \bar{\alpha})^{j-1} f\right]_*,$$

so, by Lemma 6, the function $K_{\alpha, j-1}$ is in $[f]_*$, and $g^{(j-1)}(\alpha) = 0$ for all g in $[f]_*^\perp$. Since this holds for any α occurring in $\{\alpha_k\}$ and any j such that α occurs at least j times in $\{\alpha_k\}$, any function g in $[f]_*^\perp$ has zeros at each α_k, with multiplicities according to the number of occurrences in $\{\alpha_k\}$. Since $\{\alpha_k\}$ is an infinite sequence contained in $\{|z| < r\}$, this implies $[f]_*^\perp = \{0\}$, so f is cyclic.

3. Finite-dimensional T^*-invariant subspaces.

THEOREM 2. *Every finite-dimensional T^*-invariant subspace is of the form*

$$\left((z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n} H(\beta)\right)^\perp$$

for some $\alpha_1, \ldots, \alpha_n$ in the open unit disk and $k_1, \ldots, k_n$ positive integers.

PROOF. Let M be a finite-dimensional T^*-invariant subspace. Then $T^*|M$ is an operator on the finite-dimensional space M, so it can be put in Jordan form. Thus M is the direct sum of invariant subspaces Y such that $T^*|Y$ has Jordan form

$$\begin{pmatrix} \bar{\alpha} & & 0 \\ 1 & & \\ 0 & 1 & \bar{\alpha} \end{pmatrix}$$

for some α in $\mathbf{C}$, and since $\|T^*\| = 1$, we have $|\alpha| < 1$. This means that Y has a basis $f_0, \ldots, f_k$ such that $(T^* - \bar{\alpha})f_0 = 0$ and $(T^* - \bar{\alpha})f_i = f_{i-1}$ for $i > 0$. I will show that

$$f_i = \sum_{j=0}^{i} \frac{1}{j!} \langle 1, f_{i-j} \rangle K_{\alpha, j}.$$

The proof is by induction on i. For any $g \in H^2(\beta)$, we have

$$\begin{aligned} \langle g, f_0 \rangle &= \left\langle \frac{g - g(\alpha)}{z - \alpha}(z - \alpha), f_0 \right\rangle + \langle g(\alpha), f_0 \rangle \\ &= \left\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \bar{\alpha})f_0 \right\rangle + \langle 1, f_0 \rangle g(\alpha) = \langle 1, f_0 \rangle g(\alpha), \end{aligned}$$

so $f_0 = \langle 1, f_0 \rangle K_{\alpha,0}$. For any $g \in H^2(\beta)$ we have

$$\langle g, f_i \rangle = \left\langle \frac{g - g(\alpha)}{z - \alpha}(z - \alpha), f_i \right\rangle + \langle g(\alpha), f_i \rangle.$$

The first term is

$$\left\langle \frac{g - g(\alpha)}{z - \alpha}, (T^* - \bar{\alpha})f_i \right\rangle = \left\langle \frac{g - g(\alpha)}{z - \alpha}, f_{i-1} \right\rangle,$$

so if

$$f_{i-1} = \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle K_{\alpha, j},$$

then it becomes

$$\begin{aligned} &\left\langle \frac{g - g(\alpha)}{z - \alpha}, \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle K_{\alpha, j} \right\rangle \\ &\quad = \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle \left(\frac{g - g(\alpha)}{z - \alpha} \right)^{(j)} (\alpha) \\ &\quad = \sum_{j=0}^{i-1} \frac{1}{j!} \langle 1, f_{(i-1)-j} \rangle g^{(j+1)}(\alpha) \\ &\quad = \sum_{j=1}^{i} \frac{1}{j!} \langle 1, f_{i-j} \rangle g^{(j)}(\alpha). \end{aligned}$$

Since the second term is $g(\alpha)\langle 1, f_i\rangle = (1/0!)\langle 1, f_{i-0}\rangle g^{(0)}(\alpha)$, we get

$$\langle g, f_i\rangle = \sum_{j=0}^{i} \frac{1}{j!}\langle 1, f_{i-j}\rangle g^{(j)}(\alpha),$$

so

$$f_i = \sum_{j=0}^{i} \frac{1}{j!}\langle 1, f_{i-j}\rangle K_{\alpha, j}.$$

Since it is possible to solve for each $K_{\alpha, i}$ in terms of the f_i's, the set $\{K_{\alpha,0}, \dots, K_{\alpha,k}\}$ is a basis for Y. Since M is the direct sum of spaces like Y, it follows that

$$M = \left((z-\alpha_1)^{k_1} \cdots (z-\alpha_n)^{k_n} H^2(\beta)\right)^{\perp}$$

for some $\alpha_1, \dots, \alpha_n$ in the open unit disk and $k_1, \dots, k_n$ positive integers. □

Using Theorem 2, Theorem 1 can be restated as follows.

THEOREM 1$'$. *If f is analytic in a neighborhood of the unit disk and f is not a linear combination of finitely many functions of the form $K_{\alpha, n}$, where $|\alpha| < 1$ and n is a nonnegative integer, then f is cyclic for T^*.*

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