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EXAMPLES IN THE THEORY OF SUFFICIENCY OF JETS

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ABSTRACT. It is shown that for a given nonnegative integer s, there exist a positive integer $r(s)$ and an $r(s)$-jet v_s with source at $O \in \mathbf{R}^3$ which is not V-sufficient in the class of $C^{r(s)+s}$-realizations and is C^0-sufficient in the class of $C^{r(s)+s+1}$-realizations. In the complex case, a jet with source at $O \in \mathbf{C}^2$ which is V-sufficient but not C^0-sufficient in the class of holomorphic realizations is constructed.

1. Introduction. Denote by $E^r(n,1)$ the set of all germs of C^r functions $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$. Two germs $f, g \in E^r(n,1)$ are said to be V-(resp. C^0-)*equivalent* if the set-germs $f^{-1}(0)$, $g^{-1}(0)$ are homeomorphic (resp. $g = f \circ \sigma$ for some local homeomorphism $\sigma\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$). We shall not distinguish between germs and representative functions. Let $J^r(n,1)$ be the set of r-jets of all germs belonging to $E^r(n,1)$. Given a jet $w \in J^r(n,1)$, a germ $f \in E^{r+s}(n,1)$, $s \geq 0$, whose *r-jet* $j^r f$ is equal to w, is said to be a C^{r+s}-*realization* of w. The jet w is called V-(resp. C^0-)*sufficient* in $E^{r+s}(n,1)$ if any two of its C^{r+s}-realizations are V-(resp. C^0-) equivalent. We shall identify r-jets with their polynomial representatives of degree not exceeding r.

Problems concerning sufficiency of jets have been studied by many authors. A number of criteria and characterizations have been found in the case $s = 0, 1$. Recall the following result.

THEOREM A [**4**, **9**, **11**]. *For any r-jet $w \in J^r(n,1)$, the following conditions are equivalent:*

(a) *w is V-sufficient in $E^r(n,1)$ (resp. $E^{r+1}(n,1)$);*

(b) *w is C^0-sufficient in $E^r(n,1)$ (resp. $E^{r+1}(n,1)$);*

(c) *there exist $c, \varepsilon, \delta > 0$ such that*

$$|\nabla w(x)| \geq c|x|^{r-1} \quad (\textit{resp. } |\nabla w(x)| \geq c|x|^{r-\delta}) \qquad \textit{for } |x| < \varepsilon.$$

It is also known that sufficiency in $E^{r+1}(n,1)$ does not imply sufficiency in $E^r(n,1)$ [**4**].

A natural problem, that of finding reasonable conditions characterizing r-jets sufficient in $E^{r+s}(n,1)$ for $s \geq 2$, seems to be complicated. The following example supports this opinion.

EXAMPLE 1.1. *Let s be a nonnegative integer and let $h_s\colon (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$ be the germ at 0 of the polynomial function $x_1^3 - 3x_1x_2^{2(s+2)+1} + x_3^3$. Put $r(s) = 2(s+3)+1$. Then the $(r(s)-1)$-jet u_s and $(r(s)+s)$-jet w_s of h_s have the following properties:*

(a) *u_s is C^0-sufficient in $E^{r(s)+s+1}(3,1)$;*

(b) *w_s is not V-sufficient in $E^{r(s)+s}(3,1)$.*

In particular, the $r(s)$-jet v_s of h_s is not V-sufficient in $E^{r(s)+s}(3,1)$ (in fact there

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exists a sequence $\{f_i\}$, $i = 1, 2, \ldots$, *of* $C^{r(s)+s}$*-realizations of* v_s *with* f_i *and* f_j *not* V*-equivalent for* $i \neq j$), *but is* C^0*-sufficient in* $E^{r(s)+s+1}(3,1)$. *Moreover,* $r(s) + s - 1 < L(v_s) < r(s) + s$, *where given a function* f *in a neighborhood of* $0 \in \mathbf{R}^n$, $L(f)$ *denotes the smallest positive real number* α *such that* $|\nabla f(x)| \geq c|x|^{\alpha}$ *in a neighborhood of* 0 *for some* $c > 0$.

We shall prove all these properties in §2. The example shows that a result analogous to Theorem A cannot be true for $s \geq 2$. In particular, conjecture 1 of [**2**], that an r-jet is V-(resp. C^0-) sufficient in $E^{r+s}(n,1)$, $s \geq 2$, if and only if it is V-(resp. C^0-) sufficient in $E^{r+1}(n,1)$, is false.

We shall also study sufficiency of complex jets. Denote by $H(n,1)$ and $J^r_{\mathbf{C}}(n,1)$ the space of all germs of holomorphic functions $(\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and the space of all complex r-jets of such germs, respectively. The notions of V- and C^0-sufficiency of complex r-jets in $H(n,1)$ can be defined in the obvious way. In the complex case, it is convenient to introduce a new concept which occupies an intermediate position between V- and C^0-sufficiency. An r-jet $w \in J^r_{\mathbf{C}}(n,1)$ is called *SV-sufficient* (*strongly V-sufficient*) in $H(n,1)$ if for any two holomorphic realizations f and g of w there exists a local homeomorphism $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ transforming $f^{-1}(0)$ onto $g^{-1}(0)$. Finally, given a germ $f \in H(n,1)$, we denote by $\mu(f)$ its *Milnor number*

$$\mu(f) = \dim_{\mathbf{C}} H(n)/\Delta(f),$$

where $H(n)$ is the $\mathbf{C}$-algebra of all germs of holomorphic functions $(\mathbf{C}^n, 0) \to \mathbf{C}$ and $\Delta(f)$ is the ideal of $H(n)$ generated by the first partial derivatives of f.

The following characterization is known.

THEOREM B [**2**]. *Given a complex r-jet* $w \in J^r_{\mathbf{C}}(n,1)$, *the following conditions are equivalent*:

(a) w *is SV-sufficient in* $H(n,1)$;

(b) w *is* C^0*-sufficient in* $H(n,1)$;

(c) *there exist* $c, \varepsilon, \delta > 0$ *such that* $|\nabla w(z)| \geq c|z|^{r-\delta}$ *for* $|z| < \varepsilon$;

(d) *for every holomorphic realization* f *of* w, $\mu(f) = \mu(w)$.

One might think that condition (a) in Theorem B could be replaced by a weaker one, that w is merely V-sufficient in $H(n,1)$. However, we have the following example which will be discussed in detail in §2.

EXAMPLE 1.2. *The jet* $w = z_1^4 - 4z_1z_2^8 \in J^9_{\mathbf{C}}(2,1)$ *is V-sufficient but is not SV-sufficient in* $H(2,1)$.

2. Proofs. A crucial part in our proofs depends on computing the Milnor number of certain germs.

A complex polynomial $\psi(z_1, \ldots, z_n)$ is said to be *quasi-homogeneous* of type $(\alpha_1, \ldots, \alpha_n)$, where α_j is a rational number, $0 < \alpha_j \leq 1/2$, if it is a $\mathbf{C}$-linear combination of monomials $z_1^{k_1} \cdots z_n^{k_n}$ with $\alpha_1 k_1 + \cdots + \alpha_n k_n = 1$. We shall use the following result of V. Arnol'd.

THEOREM C [**1**, THEOREM 3.1, COROLLARY 4.8]. *Suppose that a germ* $\varphi \in H(n,1)$ *is in the form*

$$\varphi = \varphi_0 + \varphi_1, \tag{2.1}$$

where φ_0 is a homogeneous polynomial of type $(\alpha_1, \dots, \alpha_n)$ with an isolated critical point at $0 \in \mathbf{C}^n$ and for any monomial $cz_1^{j_1} \cdots z_n^{j_n}$, $c \neq 0$, in the Taylor expansion of φ_1, $\alpha_1 j_1 + \cdots + \alpha_n j_n > 1$. Then

$$\mu(\varphi) = \mu(\varphi_0) = (1/\alpha_1 - 1) \cdots (1/\alpha_n - 1).$$

Consider Example 1.1. (a) Denote by ψ the complexification of h_s. Let $\varphi \in H(3,1)$ be a germ of a holomorphic function with the same $(r(s)-1)$-jet as ψ. Then φ can be written as

$$\varphi = \psi + a_1 z_2^{r(s)} + \cdots + a_{s+1} z_2^{r(s)+s} + \gamma,$$

where $a_j \in \mathbf{C}$, $\gamma \in H(3,1)$, the $(r(s)-1)$-jet of γ vanishes, and the Taylor expansion of γ does not contain terms of the form bz_2^k, $b \neq 0$, $k \leq r(s) + s$.

Two cases can occur.

(i) If $a_1 = \cdots = a_{s+1} = 0$, then $\varphi = \varphi_0 + \varphi_1$ is in the form (2.1), where $\varphi_0 = \psi$ is a quasi-homogeneous polynomial of type $(1/3, 2/(6s+15), 1/3)$, $\varphi_1 = \gamma$ and $\mu(\varphi) = \mu(\varphi_0) = 2(6s+13)$.

(ii) If $a_1 = \cdots = a_k = 0$ and $a_{k+1} \neq 0$ for some k, $0 \leq k \leq s$, then $\varphi = \varphi_0 + \varphi_1$ is in the form (2.1), where $\varphi_0 = z_1^3 + a_{k+1} z_2^{r(s)+k} + z_3^3$ is a quasi-homogeneous polynomial of type $(1/3, 1/(r(s)+k), 1/3)$, $\varphi_1 = a_{k+2} z_2^{r(s)+k+1} + \cdots + a_{s+1} z_2^{r(s)+s} - 3z_1 z_2^{r(s)} + \gamma$ and $\mu(\varphi) = \mu(\varphi_0) = 4(2(s+3)+k)$.

Note that in both cases $\mu(\varphi) = \mu(j^{r(s)+s}\varphi)$. Hence, by Theorem B, there exist $c, \varepsilon, \delta > 0$ such that

$$|\nabla\varphi(z)| \geq c|z|^{r(s)+s-\delta} \quad \text{for } |z| < \varepsilon. \tag{2.2}$$

Now let f be a $C^{r(s)+s+1}$-realization of u_s. We shall show that f is a topological projection; i.e., $(f \circ \sigma)(x_1, x_2, x_3) = x_1$ for some local homeomorphism $\sigma\colon (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$. It follows from Theorem A and (2.2) that the $(r(s)+s)$-jet of f is C^0-sufficient in $E^{r(s)+s+1}(3,1)$. Thus we may assume that f is a polynomial function. The complexification φ of f can be written as $\varphi = \varphi_0 + \varphi_1$, where φ_0 and φ_1 are as in (i) or (ii). Consider the family of germs $G_t = \varphi_0 + t\varphi_1$, $t \in [0,1]$. By Theorem C, $\mu(G_t) = \mu(\varphi_0)$ and hence there exists a neighborhood U of 0 in $\mathbf{C}^3$ such that for each t in $[0,1]$, 0 is the only critical point of G_t in U. Observe that the restriction F_t of G_t to $U \cap \mathbf{R}^3$ is a real analytic function and $0 \in \mathbf{R}^3$ is its unique critical point. By King's theorem [**7**, Theorem 1, Corollary 1] the family of germs $F_t\colon (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$ is topologically trivial; i.e., $F_t \circ \sigma_t = F_0$ for some continuous family of local homeomorphisms $\sigma_t\colon (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$. In particular, $f \circ \sigma_1 = F_0$. Thus $f \circ \sigma_1 = h_s$ if φ is in the form (i), and $f \circ \sigma_1 = x_1^3 + a_{k+1} x_2^{r(s)+k} + x_3^3$ if φ is in the form (ii). In both cases $f \circ \sigma_1 \circ \tau$ is a germ of a real analytic function, where $\tau\colon (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ is the local homeomorphism defined by $\tau(x_1, x_2, x_3) = (x_1, x_2, x_3^{1/3})$. Clearly, $0 \in \mathbf{R}^3$ is a regular point of $f \circ \sigma_1 \circ \tau$, and hence f is a topological projection.

(b) The germ $f = h_s + 2x_2^{r(s)+s+1/2}$ is a $C^{r(s)+s}$-realization of w_s and

$$(\nabla f)(x_1, x_2, x_3) = 0 \quad \text{for } x_1 = x_2^{s+2+1/2}, x_3 = 0.$$

Hence, by Theorem A, the jet w_s is not V-sufficient in $E^{r(s)+s}(3,1)$.

From (a) and (b), it is clear that the jet v_s is not V-sufficient in $E^{r(s)+s}(3,1)$ and is C^0-sufficient in $E^{r(s)+s+1}(3,1)$. Theorem A implies that $r(s)+s-1 < L(v_s) < r(s)+s$. By a theorem of Bochnak and Kuo [**3**] there exists a sequence $\{f_i\}$, $i=1,2,\ldots$, of $C^{r(s)+s}$-realizations of w_s such that f_i and f_j are not V-equivalent for $i \neq j$. Of course, each f_i is also a $C^{r(s)+s}$-realization of v_s.

Consider Example 1.2. Let φ be a holomorphic realization of w. Using essentially the same techniques as in the proof of condition (a) of Example 1.1, one can find a local homeomorphism $\sigma\colon (\mathbf{C}^2,0) \to (\mathbf{C}^2,0)$ such that $\varphi \circ \sigma = \psi_1$, where $\psi_1 = w$ or $\varphi \circ \sigma = \psi_2$, where $\psi_2 = z_1^4 - z_2^{10}$. Note that the germs at 0 of $\psi_1^{-1}(0)$ and $\psi_2^{-1}(0)$ are homeomorphic to the germ at 0 of the set $\{(z_1,z_2) \in \mathbf{C}^2 \mid z_1 z_2 = 0\}$. This means that w is V-sufficient in $H(2,1)$. Since $\mu(w) = 29$ and $\mu(w + z_2^{10}) = 27$, w is not SV-sufficient.

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