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ON THE EXISTENCE OF GREEN'S FUNCTION IN RIEMANNIAN MANIFOLDS

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ABSTRACT. This note provides a sufficient condition of geometric character for the existence of Green's function in an arbitrary complete Riemannian manifold.

1. The purpose of this note is to present a sufficient condition for an arbitrary complete C^∞ Riemannian manifold M to possess Green's function.

The condition is given in terms of some isoperimetric inequalities which we now describe.

Define $\varphi_M(t)$ by

$$\varphi_M(t) = \inf\left\{ A(\partial\Omega): \begin{array}{l} \Omega, \text{ smooth relatively compact domain} \\ \text{in } M \text{ with volume} \geqslant t, \end{array} \right\}$$

where A denotes the $((\dim M) - 1)$-dimensional measure induced by the metric of M. In particular $\varphi_M(V(\Omega)) \leqslant A(\partial\Omega)$ for each such Ω.

In §2 we will prove:

THEOREM. *If* $\int^{V(M)} dt/\varphi_M(t)^2 < \infty$, *then* M *has a Green's function.*

Recently, Dodziuk has proved in **[D]** that if $\varphi_M(t) \geqslant ct$, i.e. if the isoperimetric inequality holds in M, then M has a Green's function (see also **[T**, p. 438]).

In **[V2]**, Varopoulos has shown by extending a classical result of Ahlfors **[A]**, that if we let $L(t) = A(\partial B(x_0, t))$ for a point $x_0 \in M$, where $B(x_0, t)$ denotes the ball around x_0 of radius t, then $\int^\infty dt/L(t) < \infty$ if M has a Green's function. To see the relation with the theorem above define $\tilde{\varphi}_M$ by taking the infimum only on balls around x_0, i.e.

$$\tilde{\varphi}_M(V(B(x_0, r))) = A(\partial B(x_0, r)).$$

But then, since $(d/dr)V(B(x_0, r)) = A(\partial B(x_0, r))$, we have that

$$\int^{V(M)} \frac{dt}{\tilde{\varphi}_M(t)^2} = \int^\infty \frac{dV(B(x_0, t))}{\tilde{\varphi}_M(V(B(x_0, t)))^2} = \int^\infty \frac{dt}{L(t)}.$$

Thus our sufficient condition is close to the Ahlfors-Varopoulos necessary condition.

We should remark that, as shown in **[V1]**, the latter is also sufficient if the Ricci curvature is (semi-)positive definite but not in general; and that Milnor **[M]** (see also **[GW]**) showed that it is necessary and sufficient for 2-dimensional models.

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Since in models the Laplacian is also rotationally invariant around the pole, one can check, following the proof of the theorem, that, for n-dimensional models, $\int^{\infty} dt/L(t) < \infty$ is necessary and 'sufficient' for the existence of Green's function.

2. Proof. Assume that there is no Green's function. Let $\{\Omega_n\}_{n=0}^{\infty}$ be a sequence of smooth domains in M such that $\overline{\Omega}_n \subset \Omega_{n+1}$, $\bigcup_{n=0}^{\infty}\Omega_n = M$ and Ω_0 is a ball so small that $A(\partial\Omega_0) \leqslant \frac{1}{2}\varphi_M(V\setminus\Omega_0))$. Let ω_n be the solution to the Dirichlet problem in $\Omega_n \setminus \overline{\Omega}_n$ with data $\omega_n = 0$ on $\partial\Omega_0$, $\omega_n = 1$ on $\partial\Omega_n$.

Since we are assuming that there is no Green's function we have that $\omega_n \downarrow 0$ uniformly on compact subsets of $M \setminus \overline{\Omega}_0$. Moreover, if N denotes the outer normal vector field on $\partial\Omega_0$, then $0 < d\omega_n(N) \downarrow 0$.

If $\|\cdot\|$ denotes the norm in TM, then by the divergence theorem and since ω_n is harmonic we have, for $0 \leqslant \delta < \varepsilon < 1$,

$$\int_{\delta<\omega_n<\varepsilon} \|\nabla\omega_n\|^2\, dV = (\varepsilon-\delta)\int_{\partial\Omega_0} \nabla\omega_n \cdot N\, dA. \tag{1}$$

By the coarea formula, see [F], we have

$$\int_{\delta<\omega_n<\varepsilon} \|\nabla\omega_n\|\, dV = \int_{\delta}^{\varepsilon} A(\omega_n = t)\, dt. \tag{2}$$

Setting $\alpha_n = \int_{\partial\Omega_0} \nabla\omega_n \cdot N\, dA$ and using the Schwartz inequality, (1) and (2), we obtain

$$\left[\int_{\delta}^{\varepsilon} A(\omega_n = t)\, dt\right]^2 \leqslant (\varepsilon-\delta)\alpha_n \cdot V(\delta<\omega_n<\varepsilon).$$

Therefore

$$\left[\frac{1}{\varepsilon-\delta}\int_{\delta}^{\varepsilon} A(\omega_n = t)\, dt\right]^2 \leqslant \alpha_n \frac{V(\delta<\omega_n<\varepsilon)}{\varepsilon-\delta}$$

and, if $V_n(\varepsilon) = V(0 < \omega_n < \varepsilon)$, we have that

$$A(\omega_n = \varepsilon)^2 \leqslant \alpha_n \frac{d}{d\varepsilon} V_n(\varepsilon).$$

Since, by assumption, $A(\omega_n = \varepsilon) + A(\omega_n = 0) \geqslant \varphi_M(V(\varepsilon))$ we obtain

$$\left[\varphi_M(V_n(\varepsilon)) - A(\omega_n = 0)\right]^2 \leqslant \alpha_n V_n'(\varepsilon). \tag{3}$$

But $\omega_n \downarrow 0$ uniformly on compact subsets of $M \setminus \overline{\Omega}_0$ which implies that $V_n(1/2) \uparrow V(M\setminus\overline{\Omega}_0)$. Thus since $\varphi_M(V_n(1/2)) > \frac{1}{2}A(\partial\Omega_0)$, we obtain

$$\tfrac{1}{4}\varphi(V_n(\varepsilon))^2 \leqslant \alpha_n V_n'(\varepsilon) \quad \text{for } \varepsilon > \tfrac{1}{2},\ n > n_0 \tag{4}$$

for some $n_0 \in \mathbf{N}$. Therefore,

$$\frac{1}{8} \leqslant \alpha_n \int_{1/2}^{1} \frac{V_n'(\varepsilon)\, d\varepsilon}{\varphi_M(V_n(\varepsilon))^2} \leqslant \alpha_n \int_{V_1(1/2)}^{V(M)} \frac{dt}{\varphi(t)^2}$$

which contradicts the fact that $\alpha_n \to 0$, and so completes the proof.

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