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COMPOSING FUNCTIONS OF BOUNDED φ-VARIATION

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ABSTRACT. Let F_n be finite-valued functions on $(-\infty,\infty)$, $F_n(0)=0$, $n=1,2,\ldots$. For $x\in\mathcal{V}_\varphi\langle a,b\rangle$, the class of functions of bounded φ-variation, the compositions $F_n(x)$ are studied. The main result of this paper is Theorem 1 stating necessary and sufficient conditions for the sequence $\operatorname{var}_\psi(F_n(x),a,b)$ to be bounded for each $x\in\mathcal{V}_\varphi\langle a,b\rangle$ (ψ denotes here another φ-function).

1. Throughout this paper, by φ-function we understand a continuous, unbounded, nondecreasing function on $\langle 0,\infty)$, with $\varphi(u)=0$ iff $u=0$. Such a function is said to satisfy condition Δ_2 (for small u) whenever $\varphi(2u)\le k\varphi(u)$ with some constant $k>0$ for $0\le u\le u_0$. We denote by X the vector space of real-valued functions on $\langle a,b\rangle$ such that $x(a)=0$.

For a given partition $\pi: a=t_0<t_1<\cdots<t_n=b$, let us form the variational sum

$$\sigma_\varphi(x,\pi)=\sum_{i=1}^{n}\varphi(|x(t_i)-x(t_{i-1})|),\qquad x\in X.$$

The number

$$\operatorname{var}_\varphi(x,a,b)=\operatorname{var}_\varphi(x)=\sup_\pi\sigma_\varphi(x,\pi),$$

where the supremum is taken over all π, is called the φ-variation of x on $\langle a,b\rangle$. The following classes of functions will be considered: $\mathcal{V}_\varphi=\{x\in X:\operatorname{var}_\varphi(x)<\infty\}$ and $\mathcal{V}_\varphi^*=\{x\in X:\operatorname{var}_\varphi(\lambda x)<\infty \text{ for some } \lambda>0\}$. We will also write $\mathcal{V}_\varphi\langle a,b\rangle$ and $\mathcal{V}_\varphi^*\langle a,b\rangle$. $\mathcal{V}_\varphi^*$ is a vector space. Whenever each element $x\in\mathcal{V}_\varphi^*$ satisfies the so-called condition B.1 [**5**, p. 50] i.e.

$$\operatorname{var}_\varphi(\lambda x)\to 0\quad\text{as }\lambda\to 0,$$

a generated norm can be defined in $\mathcal{V}_\varphi^*$,

$$\|x\|_\varphi^v=\inf\{\varepsilon>0:\operatorname{var}_\varphi(x/\varepsilon)\le\varepsilon\},$$

and then $\mathcal{V}_\varphi^*$ is complete in this F-norm. When φ is a φ-function of the form $\varphi(u)=\psi(u^s)$, $0<s\le 1$, where ψ is a convex φ-function, then condition B.1 is satisfied for each $x\in\mathcal{V}_\varphi^*$. In this case, one can define in $\mathcal{V}_\varphi^*$, along with the generated norm, an s-homogeneous norm

$$\|x\|_{s,\varphi}^v=\inf\{\varepsilon>0:\operatorname{var}_\varphi(x/\varepsilon^{1/s})\le 1\};$$

$\|\cdot\|_{s,\varphi}^v$ and $\|\cdot\|_\varphi^v$ are equivalent.

Recall that the notion of functions of bounded φ-variation appeared first in the papers of Wiener [**6**] (for $\varphi(u)=u^2$), J. Marcinkiewicz [**3**] and L. C. Young [**7**] (for

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$\varphi(u) = u^p,\ p \geq 1$). The last author introduced this idea for an unrestricted φ-function [**8**]. The spaces of functions of bounded φ-variation were studied from the point of view of fundamental notions of functional analysis and some applications by J. Musielak and W. Orlicz [**4**] and R. Leśniewicz and W. Orlicz [**2**].

2. THEOREM 1. *Let φ be an arbitrary φ-function, ψ a φ-function satisfying Δ_2 for small u. Let F_n be functions on $(-\infty,\infty)$, $F_n(0) = 0$, $n = 1,2,\ldots$, such that*

(a) $\sup_n \operatorname{var}_\psi(F_n(x), a, b) \leq \infty$ *for* $x \in \mathcal{V}_\varphi\langle a, b\rangle$.

Then for every $v > 0$ there exists a constant K_v such that the inequality

(b) $\psi(|F_n(u_2) - F_n(u_1)|) \leq K_v\varphi(|u_2 - u_1|)$ *holds for* $u_2, u_1 \in \langle -v, v\rangle$, $n = 1,2,\ldots$. *Conversely,* (b) *implies* (a).

PROOF. To show (a)$\Rightarrow$(b), let us first observe that (a) implies the uniform boundedness in common of $F_n(u)$ in $\langle -v, v\rangle$. If the sequence $(F_n(u))$ were not uniformly bounded in common in $\langle -v, v\rangle$, then for some sequence of indices n_i there would exist $u_i \in \langle -v, v\rangle$ and u_0, such that

$$F_{n_i}(u_i) \to \infty, \tag{1}$$

and

$$\varphi(2|u_i - u_0|) \leq 1/2^i, \qquad i = 1,2,\ldots. \tag{2}$$

We can additionally assume $u_0 \leq u_i$. Choose arbitrarily points $a < t_0 < t_1 < t_2 < \cdots < t_n < \cdots < b$ and define the function x by taking $x(a) = 0,\ x(t_0) = u_0,\ x(t_i) = u_i,\ x(t) = u_0$ in the remaining points of $\langle a, b\rangle$. Since for $j > i$, $\varphi(|u_i - u_j|) \leq 1/2^{i-1}$, we can easily calculate by (2) that $\operatorname{var}_\varphi(x) < \infty$. Since $|F_{n_i}(x(t_i))| = |F_{n_i}(u_i)|$, then in view of (1), we have a contradiction with (a).

If condition (b) is not satisfied, then there exist indices n_i (not necessarily all different, they may even be almost all equal) and intervals $\langle u_i, v_i\rangle \subset \langle -v, v\rangle$ such that

$$k_i = \frac{\psi(|F_{n_i}(v_i) - F_{n_i}(u_i)|)}{\varphi(v_i - u_i)} \to \infty. \tag{3}$$

Since F_{n_i} are uniformly bounded in common in $\langle -v, v\rangle$,

$$v_i - u_i \to 0. \tag{4}$$

Choose $w_i = (u_i + v_i)/2$. Passing, if necessary, to a partial sequence one can assume $w_i \to w_0$. Let us consider both possible cases.

1°. w_0 is contained in an at most finite number of intervals $\langle u_i, v_i\rangle$. Assume, for instance, that infinitely many intervals not containing w_0 lie to the right of w_0. The reasoning would be analogous if infinitely many of them were to the left of w_0. Let us define by induction a partial sequence of these intervals $\langle u_i', v_i'\rangle$ having the following properties:

(α) $w_0 < u_{i+1}' < v_{i+1}' < u_i' < v_i'$, $i = 1,2,\ldots$,

(β) $\varphi(w_0 - v_i') \leq 1/2^i$, $i = 1,2,\ldots$,

(γ) $k_i' > 2^{2i+1}$, $i = 1,2,\ldots$.

Here k_i' means the k_i corresponding to the interval $\langle v_i', u_i'\rangle$ in (3), and n_i is replaced by n_i'. From (β) it follows that $\varphi(v_i' - u_i') \le 1/2^i$. We determine integers m_i so that

$$\frac{1}{2^{i+1}} < m_i\varphi(v_i' - u_i') \le \frac{2}{2^i}, \qquad i = 1, 2, \dots, \tag{5}$$

and groups of points in $\langle a, b\rangle$ in such a way that

$$\begin{aligned} &\bar{t}_1^{i+1} < t_1^{i+1} < \bar{t}_2^{i+1} < t_2^{i+1} < \cdots < \bar{t}_{m_{i+1}}^{i+1} < t_{m_{i+1}}^{i+1} \\ &\quad < \bar{t}_1^i < t_1^i < \bar{t}_2^i < t_2^i < \cdots < \bar{t}_{m_i}^i < t_{m_i}^i, \\ &\qquad\qquad i = 1, 2, \dots, t_{m_i}^i \to a,\ \bar{t}_{m_i}^i > a. \end{aligned} \tag{T}$$

Define the function

$$x(t) = \begin{cases} 0 & \text{for } t = a, \\ v_i' & \text{for } t_1^i, t_2^i, \dots, t_{m_i}^i,\ i = 1, 2, \dots, \\ u_i' & \text{for } \bar{t}_1^i, \bar{t}_2^i, \dots, \bar{t}_{m_i}^i,\ i = 1, 2, \dots, \\ v_i' & \text{for } \bar{t}_j^i < t < t_j^i,\ t_j^i < t < \bar{t}_{j+1}^i,\ j = 1, 2, \dots, m_i - 1, \\ w_0 & \text{for } t_{m_i}^i < t < \bar{t}_1^{i-1},\ t_{m_1}^1 < t \le b,\ i = 1, 2, \dots. \end{cases}$$

Both (β) and (5) imply $\mathrm{var}_\varphi(x) < \infty$. However, taking into consideration (5) and (γ), we have

$$\begin{aligned} &\psi(|F_{n_i'}(x(\bar{t}_1^i)) - F_{n_i'}(x(t_1^i))|) + \psi(|F_{n_i'}(x(\bar{t}_2^i)) - F_{n_i'}(x(t_2^i))|) + \cdots \\ &\quad + \psi(|F_{n_i'}(x(\bar{t}_{m_i}^i)) - F_{n_i'}(x(t_{m_i}^i))|) \\ &= m_i\psi(|F_{n_i'}(v_i') - F_{n_i'}(u_i')|) = k_i' m_i \varphi(v_i' - u_i') \\ &\ge 2^{2i+1}\frac{1}{2^{i+1}} = 2^i \to \infty \quad \text{as } i \to \infty. \end{aligned}$$

Hence $\mathrm{var}_\psi(F_{n_i'}(x)) \to \infty$ and we have a contradiction.

2°. w_0 is contained in infinitely many $\langle u_i, v_i\rangle$. Let $u_i < w_0 < v_i$ for some i. Then

$$\begin{aligned} S_i &= \sup\left(\frac{\psi(|F_{n_i}(v_i) - F_{n_i}(w_0)|)}{\varphi(v_i - w_0)}, \frac{\psi(|F_{n_i}(w_0) - F_{n_i}(u_i)|)}{\varphi(w_0 - u_i)}\right) \\ &\ge \frac{1}{2k}\frac{\psi(|F_{n_i}(v_i) - F_{n_i}(u_i)|)}{\varphi(v_i - u_i)}, \end{aligned} \tag{6}$$

where k is a constant such that $\psi(2u) \le k\psi(u)$, when $0 \le u \le 2c$, with $|F_{n_i}(u)| \le c$ for $-v \le u \le v$, $i = 1, 2, \dots$. This is so because the following inequalities hold:

$$\begin{aligned} \psi(|F_{n_i}(v_i) - F_{n_i}(u_i)|) &\le \psi(2|F_{n_i}(v_i) - F_{n_i}(w_0)|) + \psi(2|F_{n_i}(w_0) - F_{n_i}(u_i)|) \\ &\le kS_i(\varphi(v_i - w_0) + \varphi(w_0 - u_i)) \le 2kS_i\varphi(v_i - u_i). \end{aligned}$$

So, making use of (3) and (6) we can exhibit an infinite sequence of intervals of the form $\langle u_i', w_0\rangle$ or $\langle w_0, v_i'\rangle$ and appropriate n_i' such that either

$$\frac{\psi(|F_{n_i'}(v_i') - F_{n_i'}(w_0)|)}{\varphi(v_i' - w_0)} \ge 2^{2i+1} \tag{7}$$

or

$$\frac{\psi(|F_{n_i'}(w_0) - F_{n_i'}(u_i')|)}{\varphi(w_0 - u_i')} \ge 2^{2i+1}, \qquad i = 1, 2, \dots. \tag{7'}$$

Assume, for instance, (7) holds. The proof of (7′) would be analogous. Passing, if necessary, to a partial sequence we can assume $\varphi(v_i' - w_0) \le 1/2^i$, $i = 1, 2, \ldots$. Define integers m_i so that

$$\frac{1}{2^{i+1}} \le m_i \varphi(v_i' - w_0) \le \frac{2}{2^i}, \qquad i = 1, 2, \ldots. \tag{8}$$

Define groups of points as in (T) and the function

$$x(t) = \begin{cases} 0 & \text{for } t = a, \\ v_i' & \text{for } t = t_1^i, t_2^i, \ldots, t_{m_i}^i, \ i = 1, 2, \ldots, \\ w_0 & \text{elsewhere in } \langle a, b \rangle. \end{cases}$$

It follows from (8) that $\operatorname{var}_\varphi(x) < \infty$. However, in view of (7) we have

$$\begin{aligned} &\psi(|F_{n_i'}(x(t_1^i)) - F_{n_i'}(x(\bar{t}_1^i))|) + \psi(|F_{n_i'}(x(t_2^i)) - F_{n_i'}(x(\bar{t}_2^i))|) + \cdots \\ &\quad + \psi(|F_{n_i'}(x(t_{m_i}^i)) - F_{n_i'}(x(\bar{t}_{m_i}^i))|) \\ &= m_i \psi(|F_{n_i'}(v_i') - F_{n_i'}(w_0)|) \ge 2^{2i+1} \frac{1}{2^{i+1}} = 2^i \to \infty. \end{aligned}$$

Hence $\sup_{n_i'} \operatorname{var}_\psi(F_{n_i'}(x)) = \infty$ and we have a contradiction. Similarly, we get a contradiction if $w_0 = u_i$ or $w_0 = v_i$ for infinitely many i.

To prove (b)⇒(a), note that if $x \in \mathcal{V}_\varphi\langle a, b \rangle$, then for some v, $x(t) \in \langle -v, v \rangle$, $t \in \langle a, b \rangle$ and for an arbitrary partition $\pi\colon a = t_0 < t_1 < \cdots < t_k = b$ we have

$$\sum_{i=1}^{k} \psi(|F_n(x(t_i)) - F_n(x(t_{i-1}))|) \le K_v \sum_{i=1}^{k} \varphi(|x(t_i) - x(t_{i-1})|),$$

so

$$\operatorname{var}_\psi(F_n(x)) \le K_v \operatorname{var}_\psi(x), \qquad n = 1, 2, \ldots. \tag{9}$$

Thus (a) holds. □

A sequence of elements $(x_n) \in \mathcal{V}_\varphi\langle a, b \rangle$ is called n-convergent (two-norm convergent) to x_0 if $x_n(t) \to x_0(t)$ uniformly in $\langle a, b \rangle$, $\operatorname{var}_\varphi(x_n) \le k$, $n = 1, 2, \ldots$. n-convergence in $\mathcal{V}_\varphi^*\langle a, b \rangle$ is defined analogously.

COROLLARY. (b) *implies the following property of equicontinuity of $F_n(x)$ with respect to n-convergence*:

If the sequence $x_n \in \mathcal{V}_\varphi\langle a, b \rangle$ is n-convergent in $\mathcal{V}_\varphi\langle a, b \rangle$ to $x = 0$, then $F_n(x_n)$ is n-convergent to 0 in $\mathcal{V}_\psi\langle a, b \rangle$.

Indeed, if $x_n(a) = 0$, $\operatorname{var}_\varphi(x_n) \le k$, $n = 1, 2, \ldots$, and $x_n(t) \to 0$ uniformly in $\langle a, b \rangle$, then the functions x_n are uniformly bounded in common in some $\langle -v, v \rangle$ and by (b) we get $\psi(|F_n(x_n(t))|) \le K_v \varphi(|x_n(t)|)$, so $F_n(x_n(t)) \to 0$ uniformly in $\langle a, b \rangle$. The second part of our assertion is, in view of (9), obvious. □

Note that it follows immediately from (b) and (9) that if the sequence (x_n) is n-convergent to x_0 in $\mathcal{V}_\varphi\langle a, b \rangle$, then the sequence $(F(x_n))$ is n-convergent to $F(x_0)$ in $\mathcal{V}_\psi\langle a, b \rangle$. From Theorem 1 it follows also that if for every $x \in \mathcal{V}_\varphi\langle a, b \rangle$, $F(x) \in \mathcal{V}_\psi\langle a, b \rangle$, then this operator is continuous with respect to modular covergence: $\operatorname{var}_\varphi(x_n - x_0) \to 0$ implies $\operatorname{var}_\psi(F(x_n) - F(x_0)) \to 0$.

3. THEOREM 2. *Let φ be a φ-function satisfying the condition Δ_2 for small u and F a function on $(-\infty, \infty)$, $F(0) = 0$.*

A. *The operator $F(x) \in \mathcal{V}_\varphi\langle a, b\rangle$ for each $x \in \mathcal{V}_\varphi\langle a, b\rangle$ if and only if for every $v > 0$, there exists a constant K_v such that*

$$(*) \qquad \varphi(|F(u_2) - F(u_1)|) \le K_v \varphi(|u_2 - u_1|) \quad \text{for } u_1, u_2 \in \langle -v, v\rangle.$$

B. *Under the additional assumption that φ is strictly increasing and φ^{-1} satisfies Δ_2 for small u, the inequality $(*)$ is equivalent to*

$$(**) \qquad |F(u_2) - F(u_1)| \le K_v |u_2 - u_1| \quad \text{for } u_1, u_2 \in \langle -v, v\rangle.$$

On the above assumptions it follows from $(**)$ that for arbitrary φ-function ψ, $x \in \mathcal{V}_\psi^*\langle a, b\rangle \Rightarrow F(x) \in \mathcal{V}_\psi^*\langle a, b\rangle$.

PROOF. Ad. A. Inequality $(*)$ is an immediate consequence of Theorem 1, if we put $F_n = F$, $n = 1, 2, \ldots$.

Ad. B. Under the additional assumptions on φ we get from $(*)$

$$|F(u_2) - F(u_1)| \le \varphi_{-1}(K_v \varphi(|u_2 - u_1|)) \le \overline{K}_v |u_2 - u_1|.$$

Let $x \in \mathcal{V}_\psi^*$, i.e., $\operatorname{var}_\psi(\lambda x) \le \infty$ for some $\lambda > 0$. For an arbitrary partition $\pi: a = t_0 < t_1 < \cdots < t_n = b$ and some v we get from $(**)$

$$\sum_{i=1}^n \psi\left(\frac{\lambda}{\overline{K}_v} |F(x(t_i)) - F(x(t_{i-1}))|\right) \le \sum_{i=1}^n \psi(\lambda |x(t_i) - x(t_{i-1})|) \le \operatorname{var}_\psi(\lambda x).$$

Hence $\operatorname{var}_\psi((\lambda/\overline{K}_v) F(x)) < \infty$; that is $F(x) \in \mathcal{V}_\psi^*$.

COROLLARY. *If for any $x \in \mathcal{V}\langle a, b\rangle$ there is $F(x) \in \mathcal{V}\langle a, b\rangle$, then for arbitrary φ-function, $x \in \mathcal{V}_\varphi^*\langle a, b\rangle$ implies $F(x) \in \mathcal{V}_\varphi^*\langle a, b\rangle$.*

This Corollary and Theorem 2 generalize, in a sense, a result of M. Josephy [**1**].

REMARK. A. If ψ is of the form $\psi(u) = \chi(u^s)$, χ convex, $0 < s \le 1$, and ψ satisfies Δ_2 for small u, then the assumption (a) of Theorem 1, $\sup_n \operatorname{var}_\psi(F_n(x)) < \infty$ for $x \in \mathcal{V}_\varphi\langle a, b\rangle$, is equivalent to $\sup_n \|F_n(x)\|_{s,\psi}^v < \infty$. This last assumption is reminiscent of the one concerning sequences of linear operators in normed spaces from the Banach-Steinhaus Uniform Boundedness Principle. From this assumption follows (b) of Theorem 1, so we have (9) and thus, if $\operatorname{var}_\varphi(x) \le 1$, then with some constant C the inequality

$$\operatorname{var}_\psi(F_n(x)) \le C \operatorname{var}_\varphi(x) \le C$$

holds for $n = 1, 2, \ldots$. Hence, taking $C \ge 1$,

$$\operatorname{var}_\psi\left(\frac{F_n(x)}{C^{1/s}}\right) \le 1,$$

that is to say $\|F_n(x)\|_{s,\psi}^v \le C$ when $\operatorname{var}_\varphi(x) \le 1$, $n = 1, 2, \ldots$. Thus we obtained a conclusion analogous to the assertion of the Banach-Steinhaus theorem.

B. Put $F_n(u) = u$, $n = 1, 2, \ldots$. In this case the condition (b) is, with some $K > 0$, $u_0 > 0$, equivalent to $\psi(u) \le K\varphi(u)$ for $0 \le u \le u_0$. This is the known necessary and sufficient condition for the inclusion $\mathcal{V}_\varphi\langle a, b\rangle \subset \mathcal{V}_\psi\langle a, b\rangle$ to hold. (However, in Theorem 1 it has been additionally assumed that ψ satisfies Δ_2 for small u.)

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