

ON WEAKLY COMPACT OPERATORS ON SPACES OF VECTOR VALUED CONTINUOUS FUNCTIONS¹

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ABSTRACT. Let K and S be compact Hausdorff spaces and θ a continuous function from K onto S . Then for any Banach space E the map $f \mapsto f \circ \theta$ isometrically embeds $C(S, E)$ as a closed subspace of $C(K, E)$. In this note we prove that when E' has the Radon-Nikodým property, every weakly compact operator on $C(S, E)$ can be lifted to a weakly compact operator on $C(K, E)$. As a consequence, we prove that the compact dispersed spaces K are characterized by the fact that $C(K, E)$ has the Dunford-Pettis property whenever E has.

For a Banach space E and a compact Hausdorff space K , $C(K, E)$ will denote the Banach space of all E -valued continuous functions on K under the supremum norm. We shall identify, as usual, the topological dual of $C(K, E)$ with the space $\text{rcabv}(\text{Bo}(K), E')$ of the regular, countably additive Borel measures on K of bounded variation, with values in the topological dual E' of E , endowed with the variation norm [3, Theorem 2.2].

The notation and terminology used and not defined can be found in [6, 10, or 11].

Let K and S be compact Hausdorff spaces, and let $\theta: K \rightarrow S$ be an onto continuous function. For any Banach space E , the map $f \mapsto J_\theta(f) = f \circ \theta$ is then a linear isometry which embeds $C(S, E)$ as a closed subspace of $C(K, E)$. The following result establishes that, in some cases, every weakly compact operator on $C(S, E)$ can be lifted to a weakly compact operator on $C(K, E)$.

THEOREM 1. *Let K , S and θ be as above, and let E be a Banach space such that E' has the Radon-Nikodým property. Then, if F is a Banach space and $U: C(S, E) \rightarrow F$ is a weakly compact operator, there exists a weakly compact operator $\bar{U}: C(K, E) \rightarrow F$ so that $\|\bar{U}\| = \|U\|$ and $\bar{U} \circ J_\theta = U$.*

PROOF. Let D be the closed unit ball of F' . Then $U'(D)$ is weakly compact in $\text{rcabv}(\text{Bo}(S), E')$, and so there is a control measure λ for $U'(D)$ [3, Proposition 3.1; 6, 1.2.4]; i.e., a positive Radon measure on S such that $\lim_{\lambda(B) \rightarrow 0} \|m(B)\| = 0$, uniformly in $m \in U'(D)$. Because E' has the Radon-Nikodým property, for each $x' \in D$ there exists a Bochner integrable density $g_{x'} \in L^1(S, \lambda, E')$ for $U'(x')$. Recalling that the map $g \mapsto m_g$, where $m_g(f) = \int \langle f, g \rangle d\lambda$ for each f in $C(S, E)$, is an isometry from $L^1(S, \lambda, E')$ into $\text{rcabv}(\text{Bo}(S), E')$, it follows that $\{g_{x'}: x' \in D\}$ is weakly compact in $L^1(S, \lambda, E')$.

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Now let μ be a positive Radon measure on K such that $\theta(\mu) = \lambda$ [11, Chapter I, Theorem 12]. The map

$$L^1(S, \lambda, E') \ni g \mapsto g \circ \theta \in L^1(K, \mu, E')$$

is a linear isometry, and so $\{g_{x'} \circ \theta: x' \in D\}$ is a weakly compact subset of $L^1(K, \mu, E')$.

For $f \in C(K, E)$ and $x' \in F'$ let us define

$$\langle \bar{U}(f), x' \rangle = \int \langle f, g_{x'} \circ \theta \rangle d\mu.$$

$\bar{U}(f)$ clearly belongs to the algebraic dual of F' . But

$$\begin{aligned} |\langle \bar{U}(f), x' \rangle| &\leq \|f\|_\infty \int \|g_{x'} \circ \theta\| d\mu = \|f\|_\infty \|U'(x')\| \\ &\leq \|U\| \|f\|_\infty \|x'\|, \end{aligned}$$

which proves that $\bar{U}(f) \in F''$, and so \bar{U} is a continuous linear operator from $C(K, E)$ to F'' with $\|\bar{U}\| \leq \|U\|$. Besides, $\bar{U} \circ J_\theta = U$, and then $\|\bar{U}\| = \|U\|$.

Let $(x'_i)_{i \in I} \subset D$ be a weak* null net. Then $(U'(x'_i))_{i \in I}$ is relatively weakly compact and weak* convergent to 0. It follows that it is weakly null, and so $(g_{x'_i} \circ \theta)_{i \in I}$ converges weakly to 0 in $L^1(K, \mu, E')$. Therefore,

$$\lim_{i \in I} \langle \bar{U}(f), x'_i \rangle = 0.$$

This shows that the restriction of $\bar{U}(f)$ to D is weak* continuous, and so, by Grothendieck's theorem [9, 3.11.4], $\bar{U}(f)$ belongs to F for every f in $C(K, E)$.

Finally, let us note that $\bar{U}'(D) = \{g_{x'} \circ \theta: x' \in D\}$ (with the canonical identification of $L^1(K, \mu, E')$ as a closed subspace of $\text{rcabv}(\text{Bo}(K), E')$), and so \bar{U}' , and consequently \bar{U} , is weakly compact.

COROLLARY. *Under the assumptions of Theorem 1, if (x'_n) is a weakly convergent sequence in $\text{rcabv}(\text{Bo}(S), E')$, there exists a weakly convergent sequence (y'_n) in $\text{rcabv}(\text{Bo}(K), E')$ such that $J'_\theta(y'_n) = x'_n$ for every n .*

PROOF. We can assume that (x'_n) is weakly null. The operator $U: C(S, E) \rightarrow c_0$ defined by $U(f) = (\langle f, x'_n \rangle)$ is then weakly compact. According to Theorem 1, there is a weakly compact operator $\bar{U}: C(K, E) \rightarrow c_0$ such that $\bar{U} \circ J_\theta = U$. If $\bar{U}(f) = (\langle f, y'_n \rangle)$ for f in $C(K, E)$, the sequence (y'_n) fulfills the requirements of the corollary.

Recall that a Banach space E is said to have the Dunford-Pettis property (D.P.P.) if every weakly compact operator on E sends weakly compact sets into norm compact ones. This property was introduced by A. Grothendieck in his important paper [8] and has been intensively studied (see [5]). The long-standing open question of whether $C(K, E)$ has the D.P.P. if E has, was answered by M. Talagrand [12], who built a Banach space T such that

- (1) T and T' have unconditional basis.
- (2) T' is a Schur space; in particular, T and T' have the D.P.P.
- (3) $C(|0, 1|, T)$ does not have the D.P.P.

However, if K is a dispersed compact space (i.e., a space which does not contain any perfect set; see [10, §5]), it is known that $C(K, E)$ has the D.P.P. if E has (see [4 and 7]). Now we can prove that this property indeed characterizes the compact dispersed spaces.

THEOREM 2. *Let K be a compact Hausdorff space. The following properties are equivalent:*

- (a) K is dispersed.
- (b) If E is a Banach space with the D.P.P., so is $C(K, E)$.
- (c) If T is the Talagrand space, $C(K, T)$ has the D.P.P.

PROOF. That (a) implies (b) is known (see [4, Theorem 4; 7, Theorem 13]) and (b) \Rightarrow (c) is obvious. Let us suppose that K is not dispersed. Then there is a continuous onto function $\theta: K \rightarrow]0, 1[$ [10, §2.4.2]. According to Talagrand's result, $C(]0, 1[, T)$ does not have the D.P.P., and so there is a Banach space F , a weakly compact operator U from $C(]0, 1[, T)$ to F and a weakly compact subset H of $C(]0, 1[, T)$ such that $U(H)$ is not norm compact. Because T' is separable, it has the Radon-Nikodým property [6, III.3.1]. By Theorem 1, there is a weakly compact operator \bar{U} from $C(K, T)$ into F such that $\bar{U} \circ J_\theta = U$. But then $J_\theta(H)$ is weakly compact and $\bar{U}(J_\theta(H)) = U(H)$ is not norm compact. This shows that $C(K, T)$ does not have the D.P.P., and concludes the proof.

ADDITIONAL REMARK. After the writing of this paper, some other extension theorems for operators on $C(S, E)$ have been obtained in [2]. By using the methods of this last paper (essentially the choice of a weak* density instead of a Bochner one), one can prove that Theorem 1 is true without assuming the Radon-Nikodým property on E' . For details, we refer to [2].

We should also mention that other characterizations of compact dispersed spaces in terms of operators on spaces of continuous vector valued functions can be seen in [1].

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