PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 1, May 1986

INFINITELY MANY KNOTS WITH THE SAME POLYNOMIAL INVARIANT

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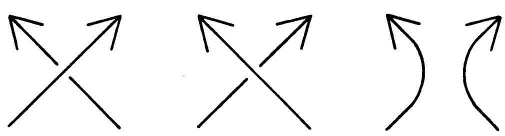
ABSTRACT. We give infinitely many examples of infinitely many knots in S^3 with the same recently discovered two-variable and Jones polynomials, but distinct Alexander module structures, which are hyperbolic, fibered, ribbon, of genus 2, and 3-bridge.

Two knots K_1 and K_2 in S^3 belong to the same isotopy type if there exists an orientation preserving homeomorphism of S^3 which maps K_1 onto K_2. We denote it by $K_1 \approx K_2$. In 1984, V. Jones **[9]** discovered a very powerful polynomial invariant of the isotopy type of an oriented knot or link. Subsequently, the Jones polynomial was generalized to the two-variable polynomial invariant simultaneously and independently by Ocneanu **[13]**, Lickorish and Millett **[12]**, Hoste **[8]**, and Freyd and Yetter. In this note we follow Lickorish and Millett. For a link L, the polynomial $L(l, m)$ is defined recursively by the following two conditions:

(I) If L_+, L_- and L_0 are three links with completely identical projections except at one crossing, where they are related as shown in Figure 1, then

$$lL_+(l,m) + l^{-1}L_-(l,m) + mL_0(l,m) = 0.$$

(II) If K is a trivial knot, then $K(l, m) = 1$.



L_+ $\quad$ L_- $\quad$ L_0

Let $\Delta_L(t)$, $\nabla_L(z)$ and $V_L(t)$ be the Alexander polynomial, the Conway polynomial **[4]** and the Jones polynomial of a link L, respectively. They can be recovered from $L(l, m)$ by the formulas

$$\Delta_L(t) = L\left(i, i(t^{1/2} - t^{-1/2})\right),$$
$$\nabla_L(t) = L(i, iz),$$
$$V_L(t) = L\left(it, i(t^{1/2} - t^{-1/2})\right),$$

where $i = \sqrt{-1}$.

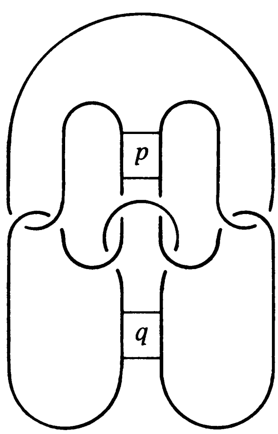
Received by the editors January 4, 1985 and, in revised form, March 1, 1985.
1980 *Mathematics Subject Classification*. Primary 57M25.
Key words and phrases. Knot, two-variable polynomial invariant, Jones polynomial, Alexander module.

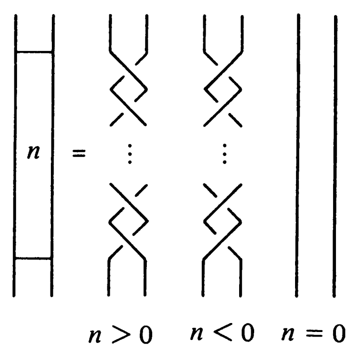
Birman [2] gives examples of a pair of distinct closed 3-braids with the same Jones and Alexander polynomials. On the other hand, it is remarked in [12], as a consequence of (I) and (II), that these polynomials are invariants of the skein equivalence class [4], (cf. [7]) of the oriented link. For example, the Kinoshita-Terasaka and the Conway 11-crossing knots with trivial Alexander polynomial have the same polynomials. Thus, using the pretzel knots [14], (cf. [1]), we have arbitrarily many distinct knots with the same polynomials.

In this note we prove

THEOREM. *There exist infinitely many examples of infinitely many knots in S^3 with the same two-variable polynomial invariant and, therefore, the same Jones polynomial, but distinct Alexander module structures, which are hyperbolic, fibered, ribbon, of genus* 2, *and* 3-*bridge*.

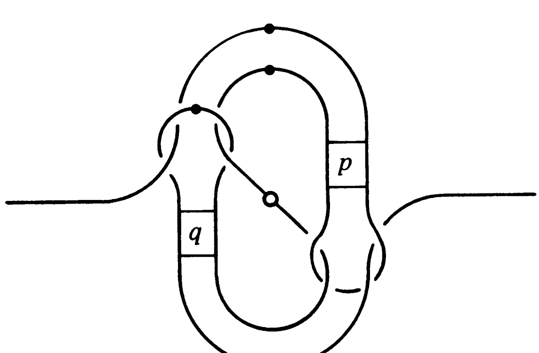
We consider the family of knots $K_{p,q}$ as shown in Figure 2, where the rectangle labeled n stands for $|n|$ full-twists as shown in Figure 3.

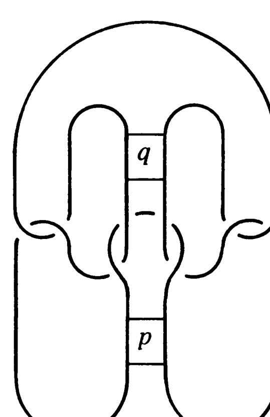




LEMMA 1. $K_{p,q} \approx K_{q,p}$.

PROOF. It is easy to see that two knots of Figures 2 and 5 are isotopic to that of Figure 4. Turning over the projection (Figure 5), we obtain that of $K_{q,p}$. □





These knots were given by the author in **[10]**, where $K_{p,p+1}$ ($p = 0, 1, 2, \ldots$) were presented as the first specific example of infinitely many distinct fibered knots with the same Alexander module structure. The following are also mentioned in **[10]**: $K_{p,q}$ are fibered of genus 2, they are the symmetric unions of the figure-eight knot wound at two places **[11]**, and, therefore, are ribbon knots. $K_{p,q}$ has the Alexander matrix

$$\begin{bmatrix} t^2 - 3t + 1 & (p-q)t \\ 0 & t^2 - 3t + 1 \end{bmatrix},$$

a presentation matrix for the Alexander module over the polynomial ring $\Lambda = Z[t, t^{-1}]$. Here we show:

LEMMA 2. *$K_{p,q}$ and $K_{p',q'}$ have the same Alexander module structure iff* $|p - q| = |p' - q'|$.

PROOF. The second elementary ideal **[6]** is $(t^2 - 3t + 1, p - q)$. The Λ-module presentation $\Lambda/(t^2 - 3t + 1, p - q)$ has the infinite presentation as an abelian group:

$$\langle \{a_i\}; \{a_{i+2} - 3a_{i+1} + a_i = (p-q)a_i = 0\} \rangle, \qquad i = 0, 1, 2, \ldots$$

which is isomorphic to $Z_{|p-q|} \oplus Z_{|p-q|}$; Z_0 means the infinite cyclic group. The result follows. □

Now we calcuate the two-variable polynomial of $K_{p,q}$. By changing one of the crossings in the p full-twists of Figure 2, we have

$$lK_{p,q}(l, m) + l^{-1}K_{p-1,q}(l, m) + m\mu = 0,$$

where $\mu = -(l + l^{-1})m^{-1}$ is the polynomial of the two-component trivial link. Then

$$\begin{aligned} K_{p,q}(l, m) - 1 &= (-l^{-2})\big(K_{p-1,q}(l, m) - 1\big) = (-l^{-2})^p\big(K_{0,q}(l, m) - 1\big) \\ &= (-l^{-2})^p\big(K_{q,0}(l, m) - 1\big) = (-l^{-2})^{p+q}\big(K_{0,0}(l, m) - 1\big). \end{aligned}$$

$K_{0,0}$ is the product of two figure-eight knots, and so

$$K_{0,0}(l, m) = (m^2 - l^2 - l^{-2} - 1)^2.$$

Thus we have

$$K_{p,q}(l, m) = (-l^{-2})^{p+q}\big((m^2 - l^2 - l^{-2} - 1)^2 - 1\big) + 1,$$

and so $K_{p,q}$ has Jones polynomial

$$(t^{-2})^{p+q}\big((t^2 - t + 1 - t^{-1} + t^{-2})^2 - 1\big) + 1.$$

Hence we obtain

LEMMA 3. *$K_{p,q}$ and $K_{p',q'}$ have the same two-variable and Jones polynomials iff* $p + q = p' + q'$.

Combining Lemmas 1–3, we can completely classify the family of knots $K_{p,q}$.

PROPOSITION. $K_{p,q} \approx K_{p',q'}$ *iff* $(p,q) = (p',q')$ *or* (q',p').

A knot K is amphicheiral if $K \approx rK$, where rK is the mirror image of K. In Figure 4, a half-rotation about a normal to the plane of projection at the origin o takes $K_{p,q}$ to $rK_{-q,-p}$, see **[18]**. Thus we have

COROLLARY. $K_{p,q}$ *is amphicheiral iff* $p + q = 0$.

LEMMA 4. $K_{p,q}$ *is a 3-bridge knot.*

PROOF. Figure 4 shows that $K_{p,q}$ has crookedness at most 3 (see [**17**, p. 115]), and so $K_{p,q}$ has bridge index at most 3. We can readily compute to obtain **[17]** that the only 2-bridge knot of genus 2 with $\Delta(-1) = \pm 25$ is $(25,9)$ in Schubert's notation, or 8_8 in the notation of Alexander and Briggs. Observing the Alexander polynomial, $K_{p,q}$ is not 8_8. This completes the proof. □

LEMMA 5. $K_{p,q}$ *is hyperbolic iff* $(p,q) \neq (0,0)$.

PROOF. Riley **[15]** observes that a 3-bridge knot is either hyperbolic, a torus knot, or a product knot. It is easy to see that the only torus knot of genus 2 is that of type $(5,2)$, and so $K_{p,q}$ is not a torus knot. It is known [**3**, 5] that the only fibered knots of genus 1 are the trefoil and figure-eight knots. If $K_{p,q}$ is not prime, then $K_{p,q}$ is the product of two figure-eight knots. Thus $K_{0,0}$ is the only product knot in this family. This completes the proof. □

Now the theorem is an immediate consequence of the lemmas.

REMARK. The invertibility of $K_{p,q}$ is unknown.

ACKNOWLEDGEMENT. I would like to thank Y. Nakanishi for showing me the proof of Lemma 2.

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