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SUBHOMOGENEOUS AF C^*-ALGEBRAS AND THEIR FUBINI PRODUCTS. II[1]

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ABSTRACT. If C is a nuclear C^*-subalgebra of a C^*-algebra A, then we have $C \otimes D = (A \otimes D) \cap (C \otimes B)$ for any C^*-algebras B and D with $D \subset B$. Using this, we show that if A and B are AF algebras and $A \otimes_F B = A \otimes B$, then either A or B must be subhomogeneous.

1. Introduction. Let A be an AF algebra. In the previous paper **[6]**, we showed that if $A \otimes_F B = A \otimes B$ for any C^*-algebra B, then A is subhomogeneous. More precisely, our arguments in **[6]** show that if A is not subhomogeneous, then $A \otimes_F B(H) \supsetneqq A \otimes B(H)$. The C^*-algebra $B(H)$ is, of course, not subhomogeneous. The purpose of this note is to show that if A and B are AF algebras neither of which are subhomogeneous, then $A \otimes_F B \supsetneqq A \otimes B$. All notation and terminology follow those of the previous paper **[6]**.

2. Intersection results for C^*-tensor products. Let A (respectively B) be a C^*-algebra with C^*-subalgebras A_1 and A_2 (respectively B_1 and B_2). Then, it is clear that

$$(A_1 \cap A_2) \otimes (B_1 \cap B_2) \subseteq (A_1 \otimes B_1) \cap (A_2 \otimes B_2). \tag{2.1}$$

Wassermann **[7]** raised the question whether the equality in (2.1) holds or not, and Huruya **[3]** and the author **[5]** gave negative answers. The following theorem gives a sufficient condition for which the equality holds in (2.1) when $A_1 \subseteq A_2$ and $B_1 \supseteq B_2$.

THEOREM 2.1. *Let A and C be C^*-algebras with $C \subset A$. Assume that the pair (A, C) satisfies the following condition*:

(2.2) *There exists a net $\{\pi_\lambda;\ \lambda \in \Lambda\}$ of completely bounded linear maps from A into C such that $\sup_\lambda\{\|\pi_\lambda\|_{CB}\} < \infty$ and $\lim_\lambda \|\pi_\lambda(x) - x\| = 0$ for $x \in C$.*

Then, we have $C \otimes D = (A \otimes D) \cap (C \otimes B)$ for any pair (B, D) of C^*-algebras with $D \subset B$.

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PROOF. Let $x \in C \otimes B$ and $\varepsilon > 0$ be given. Then, there exists an element $\sum_{i=1}^{n} a_i \otimes b_i$ in the algebraic tensor product $C \odot B$ of C and B such that

$$\left\| x - \sum_{i=1}^{n} a_i \otimes b_i \right\| < \varepsilon/2(M+1),$$

where $M = \sup_\lambda\{\|\pi_\lambda\|_{CB}\}$. Now, since $\pi_\lambda \otimes 1\colon A \otimes B \to C \otimes B$ is a bounded linear map with $\|\pi_\lambda \otimes 1\| \leqslant M$, we have

$$\|(\pi_\lambda \otimes 1)(x) - x\| \leqslant \|\pi_\lambda \otimes 1\| \left\| x - \sum_{i=1}^{n} a_i \otimes b_i \right\| + \sum_{i=1}^{n} \|\pi_\lambda(a_i) - a_i\| \, \|b_i\| + \left\| \sum_{i=1}^{n} a_i \otimes b_i - x \right\|. \tag{2.3}$$

If we choose λ_0 such that $\lambda \geqslant \lambda_0$ implies

$$\|\pi_\lambda(a_i) - a_i\| < \varepsilon/2n\|b_i\| \quad \text{for } i = 1, 2, \ldots, n,$$

then we have

$$\|(\pi_\lambda \otimes 1)(x) - x\| < M\frac{\varepsilon}{2(M+1)} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2(M+1)} = \varepsilon$$

for $\lambda \geqslant \lambda_0$ from (2.3).

Now, if $x \in (C \otimes B) \cap (A \otimes D)$, then $x \in C \otimes B$ implies $x = \lim_\lambda(\pi_\lambda \otimes 1)(x)$ from the preceding paragraph, and $x \in A \otimes D$ implies that $(\pi_\lambda \otimes 1)(x) \in C \otimes D$ because π_λ maps from A into C. Hence, we have

$$x = \lim_\lambda (\pi_\lambda \otimes 1)(x) \in C \otimes D.$$

COROLLARY 2.2. *If C is a nuclear C^*-subalgebra of a C^*-algebra A, then we have $C \otimes D = (A \otimes D) \cap (C \otimes B)$ for any pair (B, D) of C^*-algebras with $D \subset B$.*

PROOF. Note that the pair (A, C) satisfies condition (2.2) as in the proof of [**1**, Corollary 1].

3. Fubini products of AF C^*-algebras. For the sake of convenience, put $M_0 = M \cap K(H) = \{(x_n);\ x_n \in M_n \text{ and } \lim_n\|x_n\| = 0\}$ (see [**6**, Example 2.1]). First, we show that $M_0 \otimes_F M_0 \supsetneqq M_0 \otimes M_0$.

EXAMPLE 3.1. We have $M_0 \otimes_F M_0 \supsetneqq M_0 \otimes M_0$. Indeed,

$$\begin{aligned} M_0 \otimes M_0 &= (B(H) \otimes M_0) \cap (M_0 \otimes B(H)) \\ &\subsetneqq F(B(H), M_0, B(H) \otimes B(H)) \cap F(M_0, B(H), B(H) \otimes B(H)) \\ &= F(M_0, M_0, B(H) \otimes B(H)) = M_0 \otimes_F M_0, \end{aligned}$$

where the first equality follows from Corollary 2.2, and the proper inclusion follows from [**4**, Example 11].

PROPOSITION 3.2. *Let A, B, C and D be nuclear C^*-algebras with $C \subset A$ and $D \subset B$. If $A \otimes_F B = A \otimes B$, then we have $C \otimes_F D = C \otimes D$.*

PROOF. Let A_0 (respectively B_0) be an injective C^*-algebra containing A (respectively B). Then we have $C \otimes_F D = F(C, D, A_0 \otimes B_0) \subset F(A, B, A_0 \otimes B_0) = A \otimes B$. Hence, it follows that

$$
\begin{aligned}
C \otimes_F D &\subset F(C, D, A \otimes B) \\
&= F(A, D, A \otimes B) \cap (C, B, A \otimes B) \\
&= (A \otimes D) \cap (C \otimes B) \qquad (3.1) \\
&= C \otimes D, \qquad (3.2)
\end{aligned}
$$

where the equality (3.1) follows from [**2**, Theorem 3.4] and the equality (3.2) follows from Corollary 2.2.

THEOREM 3.3. *Let A and B be AF C^*-algebras. Then the following are equivalent*:
(i) $A \otimes_F B = A \otimes B$.
(ii) *Either A or B is subhomogeneous.*

PROOF. It suffices to show the implication (i) $\Rightarrow$ (ii). Assume that neither A nor B are subhomogeneous. Then, A (respectively B) contains a C^*-subalgebra C (respectively D) which is isomorphic to the C^*-algebra M_0 by [**6**,Theorem 2.3]. Now, we have a contradiction by Example 3.1 and Proposition 3.2.

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