PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 2, June 1986

EXISTENCE OF BEST n-CONVEX APPROXIMATIONS

D. ZWICK

ABSTRACT. We prove that every function f, continuous on a compact interval $[a, b]$, has a continuous, best n-convex approximation with respect to the uniform norm on $[a, b]$.

Introduction. A real-valued function g defined on a real interval I is called n-convex if its nth order divided differences $[x_0, \ldots, x_n]g$ are nonnegative for all distinct $x_0, \ldots, x_n$ in I, or equivalently if the 'augmented Vandermonde'

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_0 & \cdots & x_n \\ \vdots & & \vdots \\ x_0^{n-1} & \cdots & x_n^{n-1} \\ g(x_0) & \cdots & g(x_n) \end{vmatrix}$$

is nonnegative for all $x_0 < \cdots < x_n$ in I. Thus a 1-convex function is increasing and a 2-convex function is convex in the usual sense. n-convex functions need not be n-times differentiable, however if $g^{(n)}$ is continuous, then g is n-convex iff $g^{(n)} \geq 0$.

For $f \in C[a, b]$, an n-convex function g is a best n-convex approximation to f if

$$\|f - g\| = \inf\{\|f - \tilde{g}\| : \tilde{g} \text{ is } n\text{-convex on } [a, b]\},$$

where $\|\cdot\|$ denotes the supremum norm on $[a, b]$. As noted below, for $n \geq 2$, any function that is n-convex on $[a, b]$ is continuous and bounded in (a, b) so that if a best n-convex approximation exists then a continuous, best n-convex approximation exists as well.

Although the subject of best n-convex approximation has been treated in a number of papers over the past fifteen years (usually as 'monotone' or 'restricted derivative' approximation), these have dealt almost exclusively with n-convex approximation by polynomials [**3**, **4**], for which an extensive theory now exists. Only recently have authors even begun to consider approximation by monotone functions in general [**1**] or by convex functions [**2**]. Except for the monotone case $n = 1$, existence of continuous, best n-convex approximations to continuous functions has yet to be demonstrated. This is the purpose of this paper.[1]

Main results. As the case $n = 1$ has been thoroughly treated in [**1**], it will be tacitly assumed hereafter that $n \geq 2$.

The first lemma follows directly from [**8**, Lemma 1.1].

Received by the editors March 19, 1985.
1980 *Mathematics Subject Classification*. Primary 41A50, 41A29; Secondary 26A51.

[1]The author has recently discovered that a similar result was proved by H. G. Burchard in his dissertation, *Interpolation and approximation by generalized convex functions*, Purdue, 19678. Burchard's approach is different and somewhat more complicated than the one taken here.

(1) LEMMA. *Let F be a collection of n-convex functions, defined on $[a,b]$ and pointwise bounded on a dense subset of $[a,b]$. Then F is uniformly bounded on compact subsets of (a,b).*

This result, and the following theorem, generalize a similar result for convex functions (see e.g. [**5**, p. 167]).

(2) DEFINITION. A family F of real-valued functions defined on $I \subset R$ is said to be *uniformly Lipschitz-continuous* if there is an $L < \infty$ such that $|f(x) - f(y)| < L|x - y|$ for all $x, y \in I$ and all $f \in F$.

(3) THEOREM. *Let $\{f_k\}$ be a sequence of n-convex functions defined on (a,b). If $\{f_k\}$ is uniformly bounded on compact subsets of (a,b) then it is uniformly Lipschitz-continuous on compact subsets of (a,b).*

PROOF. Suppose that $\{f_k\}$ is not uniformly Lipschitz-continuous on $[c,d] \subset (a,b)$. Then there is a subsequence, which we relabel as $\{f_k\}$, and sequences $\{x_k\}$ and $\{y_k\}$ with the properties $c \leq x_k < y_k \leq d$, $x_k \to x$ and $y_k \to x \in [c,d]$, and

$$\left|\frac{f_k(y_k) - f_k(x_k)}{y_k - x_k}\right| \to \infty.$$

Without loss of generality the sequence of difference quotients

$$(*) \qquad \frac{f_k(y_k) - f_k(x_k)}{y_k - x_k}$$

converges either to $+\infty$ or to $-\infty$. In the latter case, we choose points $a < t_0 < \cdots < t_{n-2} < x$, and suppose that k is large enough so that $t_{n-2} < x_k < y_k$. Then, by the n-convexity of f_k,

$$D_k = \begin{vmatrix} 1 & \cdots & 1 & 1 & 1 \\ t_0 & \cdots & t_{n-2} & x_k & y_k \\ \vdots & & \vdots & \vdots & \vdots \\ t_0^{n-1} & \cdots & t_{n-2}^{n-1} & x_k^{n-1} & y_k^{n-1} \\ f_k(t_0) & \cdots & f_k(t_{n-2}) & f_k(x_k) & f_k(y_k) \end{vmatrix} \geq 0.$$

This determinant is unchanged if we subtract from each column but the first its predecessor, and then divide every element in a column (for each but the first column) by the element in its second row. We are then left with

$$D_k = c_k \cdot \begin{vmatrix} 1 & \cdots & 1 & 1 \\ [t_0, t_1]t^2 & \cdots & [t_{n-2}, x_k]t^2 & [x_k, y_k]t^2 \\ \vdots & & \vdots & \vdots \\ [t_0, t_1]t^{n-1} & \cdots & [t_{n-2}, x_k]t^{n-1} & [x_k, y_k]t^{n-1} \\ [t_0, t_1]f_k & \cdots & [t_{n-2}, x_k]f_k & [x_k, y_k]f_k \end{vmatrix},$$

where $c_k = \prod_{i=0}^{n-3}(t_{i+1} - t_i) \cdot (x_k - t_{n-2}) \cdot (y_k - x_k) > 0$, and $[x,y]f$ is the usual difference quotient. We thus have $D_k/c_k \geq 0$ for large enough k. If we now expand D_k/c_k and use the uniform boundedness of $\{f_k\}$ we get

$$0 \leq D_k/c_k = [x_k, y_k]f_k \cdot V_k + O(1),$$

where $\{V_k\}$ is a sequence of determinants converging to a positive multiple of the Vandermonde determinant $V(t_0, \ldots, t_{n-2}, x) > 0$. Since by assumption $[x_k, y_k]f_k \to -\infty$ we have arrived at a contradiction.

If the difference quotients $(*)$ converge to $+\infty$ then we choose points $a < t_0 < \cdots < t_{n-3} < x \leq d < t_n < b$ and proceed as in the previous case. Here, for large enough k, $0 \leq D_k/c_k = -[x_k, y_k]f_k \cdot V_k + O(1)$, where the V_k's converge to a positive multiple of the Vandermonde $V(t_0, \ldots, t_{n-3}, x, t_n)$. Again we arrive at a contradiction and hence $\{f_k\}$ must be uniformly Lipschitz-continuous on $[c, d]$. □

By setting $f_k \equiv f$ for each k in the proof of (3) we get

(4) COROLLARY. *If f is n-convex on $[a, b]$ then f is Lipschitz-continuous on compact subsets of (a, b).*

The following theorem extends an important convergence result, known to hold for convex functions [**5**, **6**], to n-convex functions in general.

(5) THEOREM. *Let $\{f_k\}$ be a sequence of n-convex functions, pointwise bounded on a dense subset of (a, b). Then there is an n-convex function f such that a subsequence of $\{f_k\}$ converges uniformly to f and compact subsets of (a, b).*

PROOF. By (1) $\{f_k\}$ is uniformly bounded on compact subsets of (a, b), hence by (3) it is uniformly Lipschitz-continuous, and thus is equicontinuous, on compact subsets of (a, b). By a corollary to the Arzela-Ascoli theorem [**7**, p. 179] a subsequence of $\{f_k\}$ converges pointwise to a continuous function f, which is therefore n-convex, and the convergence is uniform on compact subsets of (a, b). □

We now prove the main result of this paper.

(6) THEOREM. *Every function $f \in C[a, b]$ has a best n-convex approximation $g \in C[a, b]$.*

PROOF. We note first that if $n \geq 2$ then an n-convex function g defined on $[a, b]$ is continuous in (a, b) and bounded [**6**], and thus if $f \in C[a, b]$ then

$$\sup_{[a,b]} |f(x) - g(x)| \geq \sup_{(a,b)} |f(x) - g(x)| = \sup_{[a,b]} |f(x) - g^*(x)|,$$

where g^* is the continuous n-convex function that agrees with g in (a, b) and satisfies $g^*(a) = g(a+)$, $g^*(b) = g(b-)$. Thus we may, and will, restrict our attention in this proof to continuous n-convex functions.

Choose a sequence of n-convex functions $\{g_k\}$ such that

$$\|f - g_k\| \downarrow E(f) = \inf\{\|f - \tilde{g}\| \colon \tilde{g} \text{ is } n\text{-convex on } [a, b]\}.$$

Since $\|g_k\| - \|f\| \leq \|f - g_k\| \leq \|f - g_1\|$, we have $\|g_k\| \leq \|f\| + \|f - g_1\|$ for all k, hence $\{g_k\}$ is uniformly bounded.

By (5) there is a subsequence, which we relabel $\{g_k\}$, and an n-convex function g such that $g_k \to g$ uniformly on compact subsets of (a, b). Moreover, g is bounded and, as noted above, may be extended to a continuous function on $[a, b]$.

We now show that g is a best n-convex approximation to f on $[a, b]$. For $0 < \varepsilon < (b - a)/2$, $g_k \to g$ uniformly on $I_\varepsilon = [a + \varepsilon, b - \varepsilon]$, hence

$$\|f - g\|_{I_\varepsilon} = \lim_{k \to \infty} \|f - g_k\|_{I_\varepsilon} \leq \lim_{k \to \infty} \|f - g_k\| = E(f),$$

where $\|f-g\|_{I_\varepsilon} = \sup\{|f(x)-g(x)|: x \in I_\varepsilon\}$. Thus, as g is n-convex, we have

$$E(f) \leq \|f-g\| = \lim_{\varepsilon \downarrow 0} \|f-g\|_{I_\varepsilon} \leq E(f),$$

so that $\|f-g\| = E(f)$, and the theorem is proved. □

REFERENCES

1. R. B. Darst and S. Sahab, *Approximation of continuous and quasi-continuous functions by monotone functions*, J. Approx. Theory **38** (1983), 9–27.
2. R. Huotari and A. D. Meyerowitz, *Best L_1-approximation by convex functions*, preprint.
3. E. Kimchi and N. Richter-Dyn, '*Restricted derivatives*' *approximation to functions with derivatives outside the range*, J. Math. Anal. Appl. **54** (1976), 786–798.
4. G. G. Lorentz and K. L. Zeller, *Monotone approximation by algebraic polynomials*, Trans. Amer. Math. Soc. **149** (1970), 1–18.
5. J. T. Marti, *Konvexe analysis*, Birkhäuser Verlag, Basel and Stuttgart, 1977.
6. A. W. Roberts and D. E. Varberg, *Convex functions*, Academic Press, New York, 1966.
7. H. L. Royden, *Real analysis*, 2nd ed., Macmillan, London, 1968.
8. D. Zwick, *Properties of functions generalized convex with respect to a WT-system*, J. Approx. Theory **33** (1981), 308–317.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VERMONT, BURLINGTON, VERMONT 05405

Current address: Institut für Angewandte Mathematik der Universität Bonn, Wegelerstrasse 6, 5300 Bonn, West Germany