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A NEW INEQUALITY FOR COMPLEX-VALUED POLYNOMIAL FUNCTIONS

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ABSTRACT. Let $f_1, f_2, \ldots, f_n \colon \mathbf{C} \to \mathbf{C}$ be complex-valued polynomial functions of degrees $d_1, d_2, \ldots, d_n$, respectively, of a complex variable z. Then

$$M_{f_1} M_{f_2} \cdots M_{f_n} \geq M_{f_1 f_2 \cdots f_n} \geq k M_{f_1} M_{f_2} \cdots M_{f_n}$$

where

$$k = \left(\sin \frac{2}{n} \frac{\pi}{8d_1}\right)^{d_1} \left(\sin \frac{2}{n} \frac{\pi}{8d_2}\right)^{d_2} \cdots \left(\sin \frac{2}{n} \frac{\pi}{8d_n}\right)^{d_n}.$$

Let $\mathbf{C}$ be the field of complex numbers and $f \colon \mathbf{C} \to \mathbf{C}$ be a complex-valued polynomial function of a complex variable z. Define $M_f = \operatorname{Max}_{|z|=1} |f(z)|$.

THEOREM. *Let $f_1, f_2, \ldots, f_n \colon \mathbf{C} \to \mathbf{C}$ be polynomial functions of degrees $d_1, d_2, \ldots, d_n$, respectively, of a complex variable z. Then*

$$M_{f_1} \cdot M_{f_2} \cdots M_{f_n} \geq M_{f_1 \cdot f_2 \cdots f_n} \geq k \cdot M_{f_1} \cdot M_{f_2} \cdots M_{f_n}$$

where

$$k = \left(\sin \frac{2}{n} \frac{\pi}{8d_1}\right)^{d_1} \left(\sin \frac{2}{n} \frac{\pi}{8d_2}\right)^{d_2} \cdots \left(\sin \frac{2}{n} \frac{\pi}{8d_n}\right)^{d_n}. \tag{1}$$

PROOF. Of course the left inequality is obvious.

Let $f \colon \mathbf{C} \to \mathbf{C}$ be an arbitrary polynomial function of degree d with leading coefficient one, among the given functions $\{f_i\}$. Then we can factor $f(z)$ in the form

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_d),$$

so that

$$\begin{aligned} M_f &= \operatorname*{Max}_{|z|=1} |f(z)| = \operatorname*{Max}_{|z|=1} |z - z_1|\,|z - z_2| \cdots |z - z_d| \\ &\leq (1 + |z_1|)(1 + |z_2|) \cdots (1 + |z_d|). \end{aligned} \tag{2}$$

We will prove that for each $\varepsilon > 0$ and $n > 0$, we have for most θ that

$$|f(e^{i\theta})| = |e^{i\theta} - z_1|\,|e^{i\theta} - z_2| \cdots |e^{i\theta} - z_d| \geq M_f \cdot \left(\sin \frac{2}{n} \frac{\pi}{8d + \varepsilon}\right)^d. \tag{3}$$

However by (2), we see that it is enough to show that

$$\frac{|e^{i\theta} - z_1|\,|e^{i\theta} - z_2| \cdots |e^{i\theta} - z_d|}{(1 + |z_1|)(1 + |z_2|) \cdots (1 + |z_d|)} \geq \left(\sin \frac{2}{n} \frac{\pi}{8d + \varepsilon}\right)^d \tag{4}$$

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for all values of θ in the complement of the union U_d of intervals on $[-\pi, \pi]$ with total length $l(U_d) \leq (2/n)8d\pi/(8d+\varepsilon)$. Let z be an arbitrary element in $\{z_i\}$ and set $z = re^{ia}$, $r > 0$, and $\phi = (2/n)\pi/(8d+\varepsilon)$. Then for θ satisfying the inequality $2\phi \leq |\theta| \leq \pi - 2\phi$, we have ($\rho = 2r/(r^2+1) < 1$)

$$\begin{aligned}\frac{|1+re^{i\theta}|^2}{(1+r)^2} &= \frac{1+r^2+2r\cos\theta}{1+r^2+2r} \geq \frac{1+r^2-2r\cos 2\phi}{1+r^2+2r} \\ &= \frac{1-(2r/(1+r^2))\cos 2\phi}{1+(2r/(1+r^2))} = \frac{1-\rho\cos 2\phi}{1+\rho} \geq \frac{1-\cos 2\phi}{2} = \sin^2\phi.\end{aligned}$$

Thus $|1+re^{i\theta}|/(1+r) \geq \sin\phi$ for all values of θ in the complement of the union V of intervals with total length $l(V) \leq 8\phi$. Therefore, on a similar set we have that

$$|e^{i\theta} - re^{ia}|/(1+|re^{ia}|) \geq \sin\phi \tag{5}$$

and so (4) follows. Applying the estimate (2) for each polynomial function f_i of degree d_i, we obtain n inequalities of the form (3) (with $f = f_1, f_2, \ldots, f_n$), with each holding for all values of θ in the complement of the union U_{d_i} of intervals with total length

$$l(U_{d_i}) \leq (2/n)8d_i\pi/(8d_i+\varepsilon),$$

respectively, for $i = 1, 2, \ldots, n$.

We add this for all n, and deduce that all n inequalities (3) hold for some

$$\beta \in [-\pi, \pi] - \bigcup_{i=1}^{n} U_{d_i}.$$

Therefore

$$\begin{aligned}M_{f_1\cdot f_2\cdots f_n} &\geq |f_1(e^{i\beta})\cdot f_2(e^{i\beta})\cdots f_n(e^{i\beta})| \\ &= |f_1(e^{i\beta})|\,|f_2(e^{i\beta})|\cdots|f_n(e^{i\beta})| \geq M_{f_1}\cdot M_{f_2}\cdots M_{f_n} \\ &\quad\cdot\left(\sin\frac{2}{n}\frac{\pi}{8d_1+\varepsilon}\right)^{d_1}\left(\sin\frac{2}{n}\frac{\pi}{8d_2+\varepsilon}\right)^{d_2}\cdots\left(\sin\frac{2}{n}\frac{\pi}{8d_n+\varepsilon}\right)^{d_n}.\end{aligned} \tag{6}$$

Taking the limit as $\varepsilon \to 0$ (6) implies

$$M_{f_1 f_2\cdots f_n} \geq k\cdot M_{f_1}M_{f_2}\cdots M_{f_n} \tag{7}$$

where k satisfies (1).

REMARK. See [**1–5**] for other results and some research problems concerning inequalities on the derivative of a complex-valued function.

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