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ON RHOADES' OPEN QUESTIONS AND SOME FIXED POINT THEOREMS FOR A CLASS OF MAPPINGS

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ABSTRACT. In this paper the concept of C-mapping and some fixed point theorems for such mappings are introduced and presented.

These results are then used to answer two questions of Rhoades.

1. Introduction. Let (X, d) be a complete metric space and T a mapping from X into X. T is called a Rhoades mapping if for any $x,\ y \in X,\ x \neq y$,

$$(1.1)\quad d(Tx, Ty) < \max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\}.$$

Concerning the existence of a fixed point for such mappings T, Rhoades **[4]** pointed out that the condition

A_1. T is continuous and $\{T^n x_0\}_{n=0}^{\infty}$ has a cluster point for some $x_0 \in X$, is needed in order to ensure that every such T possesses a fixed point.

The following open questions were raised by Rhoades **[4]** (also see **[5]**).

(1) If T is a Rhoades mapping satisfying condition A_1, does T possess a fixed point?

(2) If the answer to (1) is no, then what additional hypotheses on T or X are needed to guarantee the existence of a fixed point?

Recently, some answers to these questions were obtained. In **[6]** it was shown that the answer to (1) is no. Using a different example, (1) was also answered in the negative in **[7]**. For (2) a partial answer is that the condition

A_2. T is continuous and X is compact, is sufficient. (See **[8**, Theorem 1].)

The purpose of this paper is to introduce the concept of a C-mapping and to establish necessary and sufficient conditions for such mappings to possess a fixed point. Using these results Rhoades' open questions are answered.

2. Definition and preliminaries. Throughout this paper we always assume that (X, d) is a complete metric space, T a self-map of X. T is called a C-mapping if for any $x \in X$ and any positive integer $n \geqslant 2$ satisfying

$$(2.1)\qquad T^i x \neq T^j x, \qquad 0 \leqslant i < j \leqslant n-1,$$

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we have

$$d(T^n x, T^i x) < \max_{1 \leqslant j \leqslant n} \{ d(T^j x, x) \}, \qquad i = 1, 2, \ldots, n-1. \tag{2.2}$$

REMARK 1. C-mappings are a class of important mappings containing a number of well-known mappings as special cases. For example, the mappings numbered (1)–(25) by Rhoades [4] are all special cases of C-mappings. In particular, the mappings listed below are special cases of a C-mapping.

(1) Banach's contraction mapping T [1]: i.e., there exists a number $h \in (0, 1)$ such that

$$d(Tx, Ty) \leqslant h\, d(x, y), \qquad \forall x, y \in X. \tag{2.3}$$

(2) Kannan's mapping T [3]: i.e., there exists a number $h \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leqslant h \{ d(x, Tx) + d(y, Ty) \}, \qquad \forall x, y \in X. \tag{2.4}$$

(3) Ciric's mapping T [2]: i.e., there exists a constant $h \in (0, 1)$ such that for any $x, y \in X$,

$$d(Tx, Ty) \leqslant h \max\{ d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \}. \tag{2.5}$$

(4) Rhoades' mapping T [4].

As is well known, mappings (1), (2), and (3) are all special cases of mapping (4). Therefore, in order to prove this statement it suffices to prove that the Rhoades mapping T is a special case of a C-mapping.

Let $x \in X$ and $n \geqslant 2$ be such that (2.1) is satisfied. If T is a Rhoades mapping, it can be shown by induction that (2.2) is true. In fact, for $n = 2$, from (1.1),

$$\begin{aligned} d(T^2 x, Tx) &< \max\{ d(Tx, x), d(Tx, T^2 x), d(x, Tx), d(Tx, Tx), d(x, T^2 x) \} \\ &= \max\{ d(Tx, x), d(T^2 x, x) \}, \end{aligned}$$

and (2.2) is satisfied.

Suppose that (2.2) is true for $n = k - 1$, $k \geqslant 3$, and denote

$$\alpha = \max_{1 \leqslant j \leqslant k-1} \{ d(T^j x, x) \}.$$

By the induction hypothesis and condition (1.1),

$$\begin{aligned} d(T^k x, T^{k-1} x) &< \max\{ d(T^{k-1} x, T^{k-2} x), d(T^{k-1} x, T^k x), \\ &\qquad d(T^{k-2} x, T^{k-1} x), d(T^{k-1} x, T^{k-1} x), d(T^{k-2} x, T^k x) \} \\ &\leqslant \max\{ \alpha, d(T^k x, T^{k-2} x) \}. \end{aligned}$$

Also by induction it can be shown that

$$d(T^k x, T^{k-i} x) < \max\{ \alpha, d(T^k x, T^{k-i-1} x) \}, \qquad i = 1, 2, \ldots, k-1.$$

Therefore

$$d(T^k x, Tx) < \max\{ \alpha, d(T^k x, x) \} = \max_{1 \leqslant j \leqslant k} \{ d(T^j x, x) \},$$

and successively,

$$d(T^k x, T^i x) < \max_{1 \leqslant j \leqslant k} \{ d(T^j x, x) \}, \qquad i = 1, 2, \ldots, k-1.$$

3. Main results.

THEOREM 1. *Let T be a C-mapping on X. Then T has a fixed point in X if and only if there exist integers m and n, $m > n \geqslant 0$, and a point $x \in X$ such that*

$$T^m x = T^n x. \tag{3.1}$$

If this condition is satisfied, then $T^n x$ is a fixed point of T.

PROOF. *Necessity*. Let $x^* \in X$ be a fixed point of T; i.e., $x^* = Tx^*$. Then (3.1) is true with $m = 1$ and $n = 0$.

Sufficiency. Suppose there exists a point $x \in X$ and two integers m, n, $m > n \geqslant 0$, such that $T^m x = T^n x$. Without loss of generality, we can assume that m is the minimal such integer satisfying $T^k x = T^n x$, $k > n$. Putting $y = T^n x$ and $p = m - n$, we have $T^p y = y$, and p is the minimal integer satisfying $T^k y = y$, $k \geqslant 1$.

To show that y is a fixed point of T, assume that it is not. Then $p \geqslant 2$, and

$$T^i y \neq T^j y \quad \text{for } 0 \leqslant i < j \leqslant p - 1.$$

Since T is a C-mapping,

$$\begin{aligned} d(y, T^i y) = d(T^p y, T^i y) &< \max_{1 \leqslant j \leqslant p} \{d(T^j y, y)\} \\ &= \max_{1 \leqslant j \leqslant p-1} \{d(T^j y, y)\}, \qquad i = 1, 2, \ldots, p-1. \end{aligned}$$

This implies that

$$\max_{1 \leqslant j \leqslant p-1} \{d(y, T^i y)\} < \max_{1 \leqslant j \leqslant p-1} \{d(T^j y, y)\},$$

which is a contradiction. Therefore $y = T^n x$ is a fixed point of T.

LEMMA. *Let T be a continuous C-mapping on X. Let x be a point in X for which $\{T^n x\}_{n=0}^{\infty}$ has a cluster point x_0. Then $T^n x_0$, $n = 0, 1, 2, \ldots$ are also cluster points of $\{T^n x\}_{n=0}^{\infty}$.*

PROOF. Since x_0 is a cluster point of $\{T^n x\}_{n=0}^{\infty}$, there exists a subsequence $\{T^{n_i} x\} \subset \{T^n x\}$ which converges to x_0. Since T is continuous,

$$\lim_{i \to \infty} T^{n_i + k} x = T^k x_0, \qquad k = 0, 1, 2, \ldots.$$

THEOREM 2. *Let T be a continuous C-mapping on X. Let x be a point in X for which $\{T^n x\}_{n=0}^{\infty}$ has a cluster point x_0. Then T has a fixed point in $\{T^n x_0\}_{n=0}^{\infty}$ if and only if one of the following conditions is satisfied*:

(i) $\{T^n x\}_{n=0}^{\infty}$ *is convergent.*

(ii) *There exists a positive integer m such that $T^m y = y$, where y is some point in $\{T^n x_0\}_{n=0}^{\infty}$.*

PROOF. *Necessity*. If $\{T^n x_0\} = \{x_0\}$, then it is obvious that $\{T^n x\}$ is convergent, and (i) is satisfied. If $\{T^n x_0\}$ is not the singleton x_0, let $y \in \{T^n x_0\}$ be a fixed point of T. Since y is a cluster point of $\{T^n x\}$, there exists a subsequence $\{T^{n_i} x\}$ converging to y. Thus

$$\lim_{i \to \infty} T^{n_i + k} x = T^k y = y, \qquad k = 0, 1, 2, \ldots,$$

and (ii) is satisfied.

Sufficiency. If (i) is satisfied, then $\{T^n x_0\}$ is the singleton x_0 and x_0 is a fixed point of T. If (ii) is satisfied, then T has a fixed point by Theorem 1.

The following results are immediate consequences of Theorems 1 and 2.

COROLLARY 1. *Let T be a Rhoades mapping on X. Then T has a fixed point in X if and only if there exist integers m and n, $m > n \geqslant 0$ and a point $x \in X$ such that $T^m x = T^n x$. (If this condition is satisfied then $T^n x$ is the unique fixed point of T.)*

COROLLARY 2. *Let T be a continuous Rhoades mapping on X. Let $\{T^n x\}_{n=0}^{\infty}$ have a cluster point x_0 for some $x \in X$. Then T has a fixed point in $\{T^n x_0\}_{n=0}^{\infty}$ if and only if one of the following conditions holds:*

(i) *$\{T^n x\}$ is convergent.*

(ii) *There exists a positive integer m such that $T^m y = y$, where y is some point of $\{T^n x_0\}_{n=0}^{\infty}$.*

REMARK 2. Corollary 2 gives an immediate answer to Rhoades' questions.

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