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ON FOUR-DIMENSIONAL h-COBORDISM

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ABSTRACT. The influence of the Poincaré Conjecture on the following problem of M. Cohen is examined.

Problem. Does there exist a four-dimensional h-cobordism with nontrivial Whitehead torsion?

It is proved (assuming the Poincaré Conjecture) that none of the following groups may be the fundamental group of a 4-dimensional h-corbordism with nontrivial Whitehead torsion: generalized quaternion groups, cyclic groups Z_8, Z_{12}, the binary octahedral group, the binary tetrahedral group.

There is a well-known procedure for constructing h-cobordisms with nontrivial Whitehead torsion. In fact, given a closed manifold M_0^{n-1}, $n-1 \geqslant 4$, then there is an h-cobordism $(W^n, M_0^{n-1}, M_1^{n-1})$ which realizes any prescribed torsion element $\tau \in \mathrm{Wh}(Z[\pi_1(M_0^{n-1})])$ (see [**4**, **10**]). The cases $n-1=2$ and $n-1=1$ are trivial because $\mathrm{Wh}(Z[\pi_1(M^2)]) = \mathrm{Wh}(Z[\pi_1(M^1)]) = 0$ (see [**12**, **4**]). Almost nothing is known for $n-1=3$, and the following intriguing problem was posed by M. Cohen (see [**1**]):

Problem. Does there exist a four-dimensional h-cobordism with nontrivial Whitehead torsion?

Note that if $\pi_1(M^3)$ is infinite then M^3 is a $K(\pi_1(M^3); 1)$-manifold (or $M^3 = S^1 \times S^2$) and there is a well-known conjecture (supported by the case of sufficiently large manifolds (see [**12**]) and Seifert manifolds (see [**6**])) that $\mathrm{Wh}(Z[\pi_1(M^3)]) = 0$. Therefore it seems to be reasonable to restrict our considerations to the case where $\pi_1(M^3)$ is a finite group. Every finite 3-manifold group can act freely on S^3 (maybe on fake S^3) and all such are classified (see, for example, [**3**]). The purpose of this note is to show the influence of the Poincaré Conjecture on the above problem. Modifying Milnor's argument from [**4**] we prove the following

THEOREM. *If the Poincaré Conjecture is true then none of the following groups may be the fundamental group of a four-dimensional h-cobordism with nontrivial Whitehead torsion*:

(a) *generalized quaternion groups*,
(b) *cyclic groups* Z_8 *and* Z_{12},
(c) *the binary octahedral group* $O(48)$,
(d) *the binary tetrahedral group* $T(24)$.

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If in addition every free action of a finite group on S^3 is equivalent to the orthogonal one, then the following groups are excluded:

(a) *all quaternion groups* $Q(4n)$,

(b) *all cyclic groups* Z_n,

(c) *the groups* $O(48)$ *and* $T(24)$,

(d) *the binary icosahedral group* $I(120)$.

PROOF. First we show how the Poincaré Conjecture eliminates the generalized quaternion groups. From the proof of this case it becomes clear that the remaining groups are eliminated too. Let M_0 be a 3-dimensional closed manifold with $\pi_1(M_0) = Q2^n$, $n \geqslant 3$, and let $(W; M_0, M_1)$ be an h-cobordism. We show that $\tau(W; M_0) = 0$. Consider the universal covering $\tilde{W}$ of W. The boundary $\partial\tilde{W}$ of $\tilde{W}$ is a disjoint union of Σ_1^3 and Σ_2^3, where $\Sigma_{1,2}^3$ is a homotopy 3-sphere; by our assumption $\Sigma_{1,2}^3 = S^3$. Therefore we have an h-cobordism $(\tilde{W}; S^3, S^3)$ with a free action of $Q2^n$. It is proved in **[11]** that every free action of $Q2^n$ on S^3 is equivalent to the orthogonal one. This implies that both manifolds $M_0 = S^3/Q2^n$ and $M_1 = S^3/Q2^n$ are Seifert manifolds and hence are diffeomorphic (see **[8]**) and, in fact, are isometric. To simplify our notation from now on we put $Q2^n = G$. Let $i\colon Z[G] \to Q[G]$ be the natural inclusion. It induces the homomorphism $i_*\colon K_1(Z[G]) \to K_1(Q[G])$; we denote the kernel of i_* by $SK_1(Z[G])$ and the image by $K'(Z[G])$. Note that $K_1(Q[G]) = K_1(Q) \oplus K_1(QR_G)$ where QR_G is defined by $QR_G = Q[G]/(\Sigma)$ and Σ denotes the sum of elements of G. Because G acts trivially on $H_*(S^3; Q)$, the Reidemeister torsion $\Delta(\cdot) \in \bar{K}_1(QR_G)/\mathrm{Im}\ G$ is defined (see **[4]**). One has (see **[4]**) the following formula (after the identification of fundamental groups):

$$\Delta(W) = \Delta(W; M_0) \cdot \Delta(M_0) = \Delta(W, M_1) \cdot \Delta(M_1).$$

It was proved in [7] that $SK_1(Z[G]) = 0$ which implies (see **[13]**) that $\mathrm{Wh}(Z[G])$ can be identified with

$$\mathrm{Wh}'(Z[G]) := \mathrm{Wh}(Z[G])/SK_1(Z[G]).$$

Therefore, $\mathrm{Wh}(Z[G])$ is a free abelian group and the standard involution $*$: $\mathrm{Wh}(Z[G]) \to \mathrm{Wh}(Z[G])$ is trivial (see **[13]**). Because $(W; M_0, M_1)$ is an h-cobordism then the Whitehead torsion $\tau(W, M_i) \in \mathrm{Wh}(Z[G])$, $i = 1, 2$, is defined. It is not difficult to see that $\Delta(W; M_i)$ is the image of $\tau(W, M_i)$, $i = 1, 2$, under the natural identification.

For, consider the following commutative diagram:

$$\begin{array}{ccc} K_1(ZG) & \to & K_1(Z) \oplus K_1(ZR_G) \\ \downarrow & & \downarrow \\ K_1(QG) & \to & K_1(Q) \oplus K_1(QR_G) \end{array}$$

Since $\mathrm{Wh}(ZG) = \bar{K}_1(ZG)/\text{torsion}$, the commutativity of the above diagram, together with the fact that $\bar{K}_1(Z)/\text{torsion} = 0$, shows that $\mathrm{Wh}(ZG)$ injects into $\bar{K}_1(QR_G)$. Therefore, by an argument analogous to that in [4] one concludes that $2\tau(W, M_0) = 0$ and, hence, $\tau(W; M_0) = 0$ because $\mathrm{Wh}(Z[G])$ is torsion free. This completes the case $G = Q2^n$. For $G = O(48)$ and $G = T(24)$, hyperelementary induction shows tht $SK_1(Z[G]) = 0$. Because the free actions of these groups on S^3 are equivalent to orthogonal actions (see [**11**]), then the desired result follows. The case $G = Z_8$ and $G = Z_{12}$ follows from [**4** and **9**, **11**]. Note that here the Atiyah-Bott result is needed. Now if there are only orthogonal free actions on S^3, then for every group G in the statement of the theorem $SK_1(Z[G]) = 0$ and this is all that is needed.

REMARK 1. It is quite probable that using this method one can eliminate some other groups too, but definitely not all. For example, $SK_1(Z[G])$ is in general nontrivial (see [7]) for groups given by $G = Z_p \times Q2^n$, p-prime.

REMARK 2. As it was mentioned at the beginning of this note for every closed manifold M^n, $n \neq 3$, and every $\tau \in \mathrm{Wh}(Z[\pi_1(M^n)])$ there exists an h-cobordism which realizes τ. From our theorem one can conclude that this is no longer the case when $n = 3$, at least if the Poincaré Conjecture is true. Namely, we have the following

COROLLARY. *If the Poincaré Conjecture is true then there are infinitely many examples which show that the four-dimensional version of the realization theorem for Whitehead torsion is false.*

PROOF. It is enough to show that $\mathrm{Wh}(Z[Q2^n]) \neq 0$ for $n \geqslant 4$. This is true and implicitly it is contained for example in [**14**]. Note, that $\mathrm{Wh}(Z[Q8]) = 0$.

REMARK 3. The above Corollary was also proved in [**2**] using slightly different methods.

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