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A GEOMETRICAL CHARACTERIZATION OF SINGLY GENERATED DOUGLAS ALGEBRAS

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ABSTRACT. If B is a Douglas algebra with $B \supsetneqq H^\infty + C$, then B is singly generated if and only if $\mathrm{ball}(B/H^\infty + C)$ has an extreme point.

Let H^∞ denote the space of boundary functions of bounded analytic functions in the open unit disk D. Let L^∞ be the space of bounded measurable functions on ∂D with respect to the normalized Lebesgue measure. A uniformly closed subalgebra between H^∞ and L^∞ is called a Douglas algebra. It is well known that $H^\infty + C$ is a Douglas algebra, where C is the space of continuous functions on ∂D. By Chang and Marshall's theorem [**2**, **7**], a Douglas algebra is generated by H^∞ and complex conjugate of some inner functions. It is an interesting problem to give a characterization of a Douglas algebra which is generated by complex conjugate of a single inner function (see [**7** and **3**, p. 398]). Such a Douglas algebra will be called singly generated. Up to now, its characterization has not been known. We shall give a geometrical characterization of a singly generated Douglas algebra. For a subset F of L^∞, we denote by $[F]$ the uniformly closed subalgebra generated by F.

THEOREM. *Let B be a Douglas algebra with $H^\infty + C \subsetneqq B$. Then the following assertions are equivalent.*

(i) *There is an inner function I such that $B = [H^\infty, \overline{I}]$.*

(ii) *There is an extreme point of* $\mathrm{ball}(B/H^\infty + C)$.

For a Banach space Y, we denote by $\mathrm{ball}(Y)$ the closed unit ball of Y. A point x in $\mathrm{ball}(Y)$ is called extreme if $\|x \pm y\| \leq 1$ and $y \in \mathrm{ball}(Y)$ imply $y = 0$. Extreme points of $\mathrm{ball}(B/H^\infty + C)$, where B is a Douglas algebra, are studied in [**6**, **9**].

For a Douglas algebra B, we write $M(B)$ as the maximal ideal space of B. We put $X = M(L^\infty)$. For a point x in $M(H^\infty + C)$, we denote by μ_x the representing measure on X for x, and by $\mathrm{supp}\, \mu_x$ the support set for μ_x. For an inner function I, we denote by $N(\overline{I})$ the weak*-closure in X of $\bigcup\{\mathrm{supp}\, \mu_x \colon \overline{I}|_{\mathrm{supp}\, \mu_x} \notin H^\infty|_{\mathrm{supp}\, \mu_x}\}$. We put $QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$. For some point x in X, $\{y \in X; f(y) = f(x) \text{ for every } f \in QC\}$ is called a QC-level set.

To show our theorem, we need some lemmas.

LEMMA 1 [**5**, THEOREM 1]. *If I is an inner function, then*
(1) *$Q \subset N(\overline{I})$ or $Q \cap N(\overline{I}) = \varnothing$ for every QC-level set Q, and*
(2) *$\overline{I}|_Q \notin H^\infty|_Q$ for every QC-level set Q with $Q \subset N(\overline{I})$.*

LEMMA 2 [**5**, PROOF OF COROLLARY 4]. *Let I_1 and I_2 be inner functions. Then $N(\overline{I}_1) \subset N(\overline{I}_2)$ if and only if $[H^\infty, \overline{I}_1] \subset [H^\infty, \overline{I}_2]$.*

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PROOF OF THE THEOREM. The fact that (i) $\Rightarrow$ (ii) is already pointed out in [9]. By [9, Lemma 1], for an inner function I there is an interpolating Blaschke product b such that $[H^\infty, \overline{I}] = [H^\infty, \overline{b}]$, and $\overline{b} + H^\infty + C$ is an extreme point of $\operatorname{ball}([H^\infty, \overline{I}]/H^\infty + C)$ by [6, Theorem 5].

To show the converse assertion, suppose that (i) is not true. Let $f \in B$ with $\|f + H^\infty + C\| = 1$. Since $H^\infty + C$ has the best approximation property [1], we may assume $\|f\| = 1$. We shall show that $f + H^\infty + C$ is not an extreme point of $\operatorname{ball}(B/H^\infty + C)$. By Chang and Marshall's theorem, we have $[H^\infty, f] = [H^\infty, \overline{I}; I$ is an inner function with $\overline{I} \in [H^\infty, f]]$. Then there is an inner function I_0 such that $\|I_0 f + H^\infty + C\| < 1$ and $\overline{I}_0 \in [H^\infty, f]$. we put $\alpha = 1 - \|I_0 f + H^\infty + C\|$, then $\alpha > 0$. We take a function h with

$$h \in H^\infty + C \quad \text{and} \quad \|I_0 f + h\| = \|I_o f + H^\infty + C\|. \tag{1}$$

Since $[H^\infty, \overline{I}_0] \subset B$, by our starting assumption there is an inner function J such that

$$[H^\infty, \overline{I}_0] \subsetneqq [H^\infty, \overline{J}] \subset B.$$

By Lemma 2, we get $N(\overline{I}_0) \subsetneqq N(\overline{J})$. By Lemma 2(1), there is a QC-level set Q such that $Q \subset N(\overline{J})$ and $Q \cap N(\overline{I}_0) = \emptyset$. Then there is a function q in QC satisfying

$$0 \le q \le 1 \quad \text{on } X, \qquad q = 1 \quad \text{on } Q \tag{2}$$

and

$$q = 0 \quad \text{on a neighborhood of } N(\overline{I}_0). \tag{3}$$

To show our assertion, it is sufficient to prove that

$$\overline{J} q \in B \quad \text{and} \quad \overline{J} q \notin H^\infty + C, \tag{4}$$

and

$$\|f \pm \alpha \overline{J} q + H^\infty + C\| \le 1. \tag{5}$$

(4) follows easily from (2) and Lemma 1(2). We shall prove that $\|f + \alpha \overline{J} q + H^\infty + C\| \le 1$; the other will be obtained by the same way. Let us take a measure μ on X satisfying

$$\|\mu\| = 1 \quad \text{and} \quad \mu \perp H^\infty + C, \tag{6}$$

that is, μ is an annihilating measure for $H^\infty + C$ having the unit total variation, and

$$\|f + \alpha \overline{J} q + H^\infty + C\| = \int_X (f + \alpha \overline{J} q)\, d\mu. \tag{7}$$

We put

$$E = \{x \in X; q(x) = 0\}. \tag{8}$$

Since $q \in QC$, E is a peak set for $H^\infty + C$. By the Glicksberg peak set theorem [3] and (6), we have $\mu|_E \perp H^\infty + C$, so that

$$\mu|_{X \setminus E} \perp H^\infty + C. \tag{9}$$

By (3), we can take $q_0 \in QC$ with $q_0 = 0$ on $N(\overline{I}_0)$ and $q_0 = 1$ on $X\backslash E$. Since $q_0\overline{I}_0|_{\operatorname{supp}\mu_x} \in H^\infty|_{\operatorname{supp}\mu_x}$ for every $x \in M(H^\infty + C)$, we have $q_0\overline{I}_0 \in H^\infty + C$ by [**8**], so that $q_0\overline{I}_0 h \in H^\infty + C$ by (1). By (9), we get

$$\int_{X\backslash E} \overline{I}_0 h\, d\mu = 0. \tag{10}$$

Then

$$\begin{aligned}
\|f + \alpha\overline{J}q + H^\infty + C\| &= \int_X (f + \alpha\overline{J}q)\, d\mu \quad \text{by (7)} \\
&= \int_E (f + \alpha\overline{J}q)\, d\mu + \int_{X\backslash E} (f + \alpha\overline{J}q)\, d\mu \\
&= \int_E f\, d\mu + \int_{X\backslash E} (f + \alpha\overline{J}q + \overline{I}_0 h)\, d\mu \quad \text{by (8) and (10)} \\
&\le \|\mu|_E\| + \|\mu|_{X\backslash E}\|\{\|f + \overline{I}_0 h\|_{X\backslash E} + \alpha\} \quad \text{by } \|f\| \le 1 \\
&\le \|\mu|_E\| + \|\mu|_{X\backslash E}\|\{\|I_0 f + h\| + \alpha\} \\
&\le 1 \quad \text{by (1) and (6).}
\end{aligned}$$

Thus we get (5) and complete the proof.

REMARK. In the proof of (ii) $\Rightarrow$ (i), we actually prove that if $f \in B$, $\|I_0 f + H^\infty + C\| < 1$ and $[H^\infty, \overline{I}_0] \subsetneqq B$ for some inner function I_0 with $\overline{I}_0 \in [H^\infty, f]$, then $f + H^\infty + C$ is not an extreme point of $\operatorname{ball}(B/H^\infty + C)$. Consequently, if $f + H^\infty + C$ is an extreme point of $\operatorname{ball}(B/H^\infty + C)$, and if I_0 is an inner function such that $\overline{I}_0 \in [H^\infty, f]$ and $\|I_0 f + H^\infty + C\| < 1$, then $B = [H^\infty, \overline{I}_0] \subset [H^\infty, f] \subset B$. Thus we get: If $f + H^\infty + C$ is an extreme point of $\operatorname{ball}(B/H^\infty + C)$, then $B = [H^\infty, f]$.

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