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ON THE HEAT KERNEL COMPARISON THEOREMS FOR MINIMAL SUBMANIFOLDS

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ABSTRACT. In [3], Cheng, Li and Yau proved comparison theorems (upper bounds) for the heat kernels on minimal submanifolds of space forms. In the present note we show that these comparison theorems together with a series of corollaries remain true for minimal submanifolds in ambient spaces with just an upper bound on the sectional curvature.

1. Introduction. Let M^m be a minimally immersed submanifold of N^n. For a given point $p \in M$ we define the *normal range* $U(p)$ to be the complement of the cut locus of p in N. Let $r_p(\cdot)$ denote the distance function from p in N. The ball $B_R(p) = \{x \in N \mid r_p(x) \leqslant R\}$ is then *regular* if $B_R(p) \subset U(p)$ and (when $\sup K_N = b > 0$) $R \leqslant \pi/2\sqrt{b}$.

Let $D \subset M^m$ be a compact domain of M containing p. Following [3] we denote the p-centered heat kernels on D by $H(p, y, t)$ (with Dirichlet boundary condition) and $K(p, y, t)$ (with Neumann boundary condition) respectively. If $\tilde{D}_R^b(\tilde{p})$ denotes the totally geodesic disc with center $\tilde{p}$, radius R and dimension m in a space form $\tilde{N}_b^n$ of constant curvature $b \in \mathbf{R}$, then the $\tilde{p}$-centered heat kernels on $\tilde{D}$ only depend on $r_{\tilde{p}}(\cdot)$ and t; hence we may, and do, write them as $\tilde{H}_R^b(r_{\tilde{p}}(y), t)$ and $\tilde{K}_R^b(r_{\tilde{p}}(y), t)$ respectively.

We can now formulate the extended comparison theorems for H and K as follows.

THEOREM 1. *Let M^m be a minimally immersed submanifold of N^n with $K_N \leqslant b$. Let D be a compact domain in M.*

(i) *If D is contained in a regular ball $B_R(p) \subset N$ for some $p \in D$, then the Dirichlet heat kernel on D satisfies*

$$H(p, y, t) \leqslant \tilde{H}_R^b\big(r_p(y), t\big) \quad \textit{for all } y \in D \textit{ and } t \in \mathbf{R}_+. \tag{1.1}$$

(ii) *If $D = B_R(p) \cap M$ for some (not necessarily regular) ball $B_R(p) \subset U(p) \subset N$, $p \in D$, and (if $b > 0$) $R \leqslant \pi/\sqrt{b}$, then the Neumann heat kernel on D satisfies*

$$K(p, y, t) \leqslant \tilde{K}_R^b\big(r_p(y), t\big) \quad \textit{for all } y \in D \textit{ and } t \in \mathbf{R}_+. \tag{1.2}$$

This type of theorem was first proved by Cheng, Li and Yau for space form ambient spaces N^n (cf. [3, Theorems 1 to 5]). The extension to ambient spaces with

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variable curvature is essentially a consequence of the following result, which may be proved by standard index comparison theory (cf. [**4**, Proposition 8]).

PROPOSITION 2. *We make the same assumptions as in Theorem* 1. *Let* $F\colon [0, R] \to \mathbf{R}$ *be a smooth function with* $F'(r) \geqslant 0$ *for all* $r \in [0, R]$, *and let* $\{X_j\}$, $1 \leqslant j \leqslant m$, *be an orthonormal basis of* T_qD. *Then the Laplacian* Δ_D *on* D *satisfies the following inequality at* q:

$$\Delta_D(F \circ r_p|_D) \geqslant \left(F''(r) - F'(r)h_b(r)\right)\sum_{j=1}^{m}\langle \operatorname{grad} r, X_j\rangle^2 + mF'(r)h_b(r), \tag{1.3}$$

where $h_b(r)$ *is the constant mean curvature of any distance sphere of radius* r *in a space form of constant curvature* b.

For the proof of Theorem 1 we also need the following special version of a result due to Cheeger and Yau [**1**, Lemma 2.3].

PROPOSITION 3.

$$(\partial/\partial r)\tilde{H}_R^b(r, t) < 0, \tag{1.4}$$

and

$$(\partial/\partial r)\tilde{K}_R^b(r, t) < 0 \tag{1.5}$$

for all $t > 0$ *and* $r \in [0, R]$ (*with* $R <$ (*rsp.* $\leqslant$) $\pi/\sqrt{b}$ *if* $b > 0$).

2. Proof of Theorem 1 and some consequences. Following [**3**] closely throughout, we only have to show that the transplanted heat kernels $\tilde{H}_R^b(r_p(y), t)$ and $\tilde{K}_R^b(r_p(y), t)\colon \{p\} \times D \times [0, \infty[\to [0, \infty]$ satisfy $\Box_y \tilde{H} \leqslant 0$ and $\Box_y \tilde{K} \leqslant 0$ respectively. Here $\Box_y = \Delta_y - \partial/\partial t$, where Δ_y is the Laplacian operating on functions on the second factor in the domain $\{p\} \times D \times [0, \infty[$.

We rewrite $\tilde{H}_R^b(r_p(y), t)$ as a function of s and t, i.e. $\tilde{H}_R^b(r, t) = \tilde{H}(s(r), t) = \tilde{H}(s, t)$, where

$$s(r) = \begin{cases} 1 - \cos(\sqrt{b}\, r) & \text{if } b > 0, \\ r^2/2 & \text{if } b = 0, \\ \cosh(\sqrt{-b}\, r) - 1 & \text{if } b < 0. \end{cases} \tag{2.1}$$

Now consider the following identity:

$$\Delta_y \tilde{H}(s, t) = \tilde{H}''\|\operatorname{grad}_D s\|^2 + \tilde{H}'\Delta_D s, \tag{2.2}$$

where $\tilde{H}' = (\partial/\partial s)\tilde{H}(s, t)$.

From Proposition 2 with $F(r) = s(r)$ and $s''(r) - h_b(r)s'(r) \equiv 0$ we get $\Delta_D s \geqslant m(ds/dr)h_b(r) = \tilde{\Delta}_{\tilde{D}} s$, where $\tilde{\Delta}_{\tilde{D}}$ is the Laplacian on the *space form* disc $\tilde{D}_R^b(\tilde{p})$. Proposition 3 implies $(ds/dr)\tilde{H}' \leqslant 0$, and since $ds/dr \geqslant 0$ we get $\tilde{H}'\Delta_D s \leqslant \tilde{H}'\tilde{\Delta}_{\tilde{D}} s$. Furthermore, $\|\operatorname{grad}_D s\| \leqslant \|\operatorname{grad}_N s\| = \|\widetilde{\operatorname{grad}}_{\tilde{D}} s\|$, and finally also $\tilde{H}'' \geqslant 0$ (by [**3**, pp. 1038–1043 and 1045–1049]). In total we therefore have from (2.2)

$$\Delta_y \tilde{H}(s, t) \leqslant \tilde{H}''\|\widetilde{\operatorname{grad}}_{\tilde{D}} s\|^2 + \tilde{H}'\tilde{\Delta}_{\tilde{D}} s = \tilde{\Delta}_y \tilde{H}(s, t), \tag{2.3}$$

so that $\Box_y \tilde{H} \leqslant \tilde{\Box}_y \tilde{H} = 0$.

The inequality $\square_y \tilde{K} \leqslant \tilde{\square}_y \tilde{K} = 0$ for the transplanted Neumann heat kernel follows similarly from $\tilde{K}'' \geqslant 0$ [**3**, pp. 1044 and 1049–1050]. The proof may now be completed by Proposition 1 of [**3**]. □

Once Theorem 1 is in hand we may now consider the series of corollaries given in [**3**] for similar extensions. Since the proofs of the generalized versions follow almost verbatim the space form proofs we will omit them.

COROLLARY A. *Let M^m be a minimally immersed submanifold of N^n. Suppose $K_N \leqslant b$ and let f be a nonnegative subharmonic function on M. If $p \in M$ and $\Omega_R = B_R(p) \cap M$ for a regular ball $B_R(p)$, then*

$$f(p) \leqslant C^{-1}(m, b, R) \int_{\partial \Omega_R} f * 1, \tag{2.4}$$

where

$$C(m, b, R) = m\omega_m \cdot \begin{cases} \left(\sqrt{b}^{-1} \sin \sqrt{b} R\right)^{m-1} & \text{if } b > 0, \\ R^{m-1} & \text{if } b = 0, \\ \left(\sqrt{-b}^{-1} \sinh \sqrt{-b} R\right)^{m-1} & \text{if } b < 0, \end{cases}$$

and ω_m is the volume of the unit m-ball in $\mathbf{R}^m$.

COROLLARY B. *Let M^m be a minimally immersed submanifold of N^n, $K_N \leqslant b$. Let $B_R(p)$ be a ball in the normal range of $p \in M$. Then*

$$\operatorname{vol}(B_R(p) \cap M) \geqslant \operatorname{vol}\left(\tilde{D}^b_{\min\{R, \pi/\sqrt{b}\}}\right). \tag{2.5}$$

In particular, if M is compact and contained in $U(p)$, then $b > 0$ and

$$\operatorname{vol}(M^m) \geqslant \operatorname{vol}(S^m_b), \tag{2.6}$$

where S^m_b is the round sphere of dimension m and constant curvature b. If equality occurs in (2.6), *and if M is contained in $U(p)$ for every $p \in M$, then*

$$\#\{i \mid 0 < \lambda_i(M) \leqslant mb\} \leqslant m + 1, \tag{2.7}$$

where $\{\lambda_i(m)\}$ is the ordered set of eigenvalues (with multiplicities) of Δ_M.

REMARKS. The last statement follows from the generalization of Theorem 6 in [**3**]. The inequality (2.6) generalizes a result of B.-Y. Chen who proved it for compact minimal submanifolds of spheres (cf. [**2**]).

COROLLARY C. *Let M^m be a minimally immersed submanifold of N with $K_N \leqslant b$. Let D be a compact domain in M which is contained in a regular ball $B_R(p)$ for some $p \in M$. Then the first Dirichlet eigenvalue $\lambda_1(D)$ of Δ_D satisfies*

$$\lambda_1(D) \geqslant \lambda_1\left(\tilde{D}^b_R\right) \geqslant m\pi^2/4R^2. \tag{2.8}$$

If the first inequality is an equality, then D is radial, i.e., D is generated by geodesics of length R from p.

Furthermore, if $b \leqslant 0$, then the kth Dirichlet eigenvalue for D satisfies

$$(2.9) \qquad (\lambda_k(D))^{m/2} \geqslant \frac{k \cdot (4\pi)^{m/2}}{e \cdot \operatorname{vol}(D)}.$$

REMARK. The inequality $\lambda_1(D) \geqslant m\pi^2/4R^2$ was proved in [**4**].

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