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NONARCHIMEDEAN $C^{\#}(X)$

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ABSTRACT. Let E be a nonarchimedean rank-one valued field, and X an ultraregular topological space. We consider the Gelfand subalgebra $C^{\#}(X,E)$ of the algebra of all E-valued continuous functions on X, and the algebra $F(X,E)$ consisting of those E-valued continuous functions f for which there exists a compact set $K \subset X$ such that $f(X-K)$ is finite. We obtain some characterizations of $C^{\#}(X,E)$, analogous to those obtained in the real case, which we use to find conditions that imply the equality $C^{\#}(X,E) = F(X,E)$ holds.

For T a completely regular topological space and $C(T,R)$ the space of continuous real valued functions on T, let $C^{\#}(T,R)$ denote the Gelfand subalgebra of $C(T,R)$, consisting of all $f \in C(T,R)$ with the property that, for every maximal ideal m of $C(T,R)$, there exists an $r \in R$ such that $(f-r) \in m$. We shall denote by $F(T,R)$ the subalgebra of $C(T,R)$ consisting of those $f \in C(T,R)$ for which there exists a compact $K \subset T$ (K depending on f) such that $f(T-K)$ is finite. The basic properties of $C^{\#}(T,R)$ are established in [**NR**, **C** and **SZ**], and they are summarized in [**H**, Theorem 2.1]. It follows from [**SZ** and **N**] that if T is a real compact and locally compact space or if T is a normal metacompact and locally compact space, then $C^{\#}(T,R) = F(T,R)$.

Now, let E be a nonarchimedean rank-one valued field (which we do not assume to be complete), and X an ultraregular topological space. Let $C(X,E), C^{\#}(X,E)$ and $F(X,E)$ stand for the nonarchimedean analogue of the concepts defined above. As the main result in [**D$_2$**] we saw, that if one assumes X is paracompact and locally compact, then $C^{\#}(X,E) = F(X,E)$. In the present paper, by using a nonarchimedean analogue of [**H**, Theorem 2.1], we obtain all the results of [**D$_2$**], with a weaker hypothesis in the case of the main result, as relatively simple corollaries.

If A is a unitary commutative ring, $\mathcal{M}(A)$ will denote the set of all maximal ideals of A endowed with the Zariski topology (or hull-kernel topology). Thus $\mathcal{M}(C(X,E)) = \beta_0 X$ (the Banaschewski compactification of X). For the rest we shall use the notation of [**BB**] except that we shall use "cl" to denote topological closure.

Let $E^\frown$ be the completion of the valued field E.

LEMMA 1. $C^{\#}(X,E) = C^{\#}(X,E^\frown) \cap C(X,E)$.

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PROOF. It suffices to take into account that, by the ultraregular analogue of the Gelfand-Kolmogoroff theorem (see [**BB**, Theorem 6]), one has a bijection (in fact a homeomorphism) $\mathcal{M}(C(X, E^\frown)) \to \mathcal{M}(C(X, E))$ sending each maximal ideal $m^\frown$ of $C(X, E^\frown)$ to $m^\frown \cap C(X, E)$.

REMARK. In dealing with $C^\#(X, E)$, sometimes this lemma can allow us to assume, without loss of generality, that E is a complete field. In particular, the referee had previously pointed out to us that something like this was necessary to close a gap in the second part of the proof of Proposition 1 of [$\mathbf{D_2}$], as the argument we use there does not make apparent that, for incomplete fields, $f^\beta \colon \mathcal{M} \to L$ does indeed carry all of $\mathcal{M}$ to L. Now the lemma does close that gap.

Let N denote the discrete space of positive integers.

LEMMA 2. *The functions in $C^\#(N, E)$ are exactly those with finite range.*

PROOF. If $f \in C(N, E)$ has a finite range $f(N) = \{\lambda_1, \ldots, \lambda_n\}$, then $\Pi(f - \lambda_i) = 0$. Hence for every maximal ideal m of $C(N, E)$ one has $\Pi(f - \lambda_i) \in m$ and so $(f - \lambda_i) \in m$ for some λ_i. Thus $f \in C^\#(N, E)$. In order to see the converse, take $f \in C(N, E)$ with $f(N)$ infinite, and for any $\lambda \in E$ set $Z_\lambda = f^{-1}(E - \{\lambda\})$. The family (Z_λ) has the finite intersection property, so there is a maximal ideal m of $C(N, E)$ such that $Z_\lambda \in Z[m]$ for every $\lambda \in E$. But for any $\lambda \in E$ one has $Z(f - \lambda) \cap Z_\lambda = \varnothing$ and hence $(f - \lambda) \notin m$, so $f \notin C^\#(N, E)$.

LEMMA 3. *Let $f \in C(X, E)$ and assume $f(X)$ is not precompact. Then there is a clopen partition $(U_i)_{i \in I}$ of X and there exist $x_i \in U_i$ such that $f(\{x_i / i \in I\})$ is infinite.*

PROOF. If $f(X)$ is not precompact, then there exists $\varepsilon > 0$ such that $f(X)$ has no finite covers by ε-radius spheres. For every $\alpha \in f(X)$ set $B(\alpha) = \{\mu \in E / |\mu - \alpha| \leq \varepsilon\}$. Since any two spheres $B(\alpha)$ are either equal or disjoint, there exist $(\alpha_i)_{i \in I}$, I being an infinite set, such that the sets $B(\alpha_i)$ are pairwise disjoint and form a clopen cover of $f(X)$. Now set $U_i = f^{-1}(B(\alpha_i))$ and choose $x_i \in X$ such that $f(x_i) = \alpha_i$.

It will be said that a subset S of X is C-embedded in X with respect to E if every continuous function from S into E has a continuous extension to X.

In analogy with the Hewitt real compactification of a completely regular space, we let $\nu_0 X$ be the set of all $p \in \beta_0 X$ such that, for every sequence (V_n) of neighbourhoods of p in $\beta_0 X$, $\bigcap V_n \cap X \neq \varnothing$.

Now we shall state the nonarchimedean analogue of [**H**, Theorem 2.1] (see also [**SZ**, **C** and **NR**]).

THEOREM 1. *If $f \in C(X, E)$, then the following are equivalent:*

(a) *$f \in C^\#(X, E)$,*

(b) *$f(D)$ is finite for every copy D of N which is C-embedded in X with respect to E,*

(c) *$f(Z)$ is compact for every E-zero-set Z in X,*

(d) *$f(X)$ is compact and for every $\lambda \in E$, $\mathrm{cl}_{\beta_0 X} Z(f - \lambda) = Z(\beta_0 f - \lambda)$,*

(d′) *$f(X)$ is relatively compact and for every $\lambda \in E$, $\mathrm{cl}_{\beta_0 X} Z(f - \lambda) = Z(\beta_0 f - \lambda)$.*

Moreover if E has nonmeasurable cardinality, the above conditions are also equivalent to

(e) *$f(X)$ is compact and, for every $p \in \beta_0 X - \nu_0 X$, there is a neighbourhood of p in $\beta_0 X$ on which $\beta_0 f$ is constant.*

PROOF. (a)$\Rightarrow$(b). The restriction map $C(X,E) \to C(D,E)$, $f \mapsto f|_D$ is a surjective E-algebra homomorphism, so if $f \in C^{\#}(X,E)$ then $f|_D \in C^{\#}(D,E)$, and hence by Lemma 2 $f(D)$ is finite.

(b)$\Rightarrow$(c). Let $Z = Z(g), g \in C(X,E)$. From Lemma 3, $f(X)$ is precompact and so $f(Z)$ is too. To show $f(Z)$ is compact, we are going to see that $f(Z) = \mathrm{cl}_{E^\frown} f(Z)$. Assume, otherwise, that there exists $\lambda \in \mathrm{cl}_{E^\frown} f(Z) - f(Z)$. Then the set $A = \{x \in X / |g(x)| < |f(x) - \lambda|\}$ is a clopen set such that $Z \subset A$ and $Z(f-\lambda) \subset X - A$, and the continuous function $1/(f-\lambda)\colon A \to E^\frown$ does not have precompact range. By Lemma 3, there is a clopen partition $(U_i)_{i\in I}$ of A and there is $x_i \in U_i$ such that $1/(f-\lambda)$ takes infinite values on the set $\{x_i / i \in I\}$, so $f(\{x_i / i \in I\})$ is infinite One immediately sees that this is contradictory to the assumption that (b) holds.

(c)$\Rightarrow$(d). Let $p \in Z(\beta_0 f - \lambda)$. By [**BB**, Theorem 6], $m_p = \{g \in C(X,E) / p \in \mathrm{cl}_{\beta_0 X} Z(g)\}$ is a maximal ideal and $h \in C(X,E)$ belongs to m_p iff $Z(h)$ meets $Z(g)$ for each g in m_p. By (c), $f(Z(g)) = \beta_0 f(\mathrm{cl}_{\beta_0 X} Z(g))$ for each g in m_p. Thus there exists $x \in Z(g)$ such that $f(x) = \beta_0 f(p) = \lambda$. Hence $Z(f-\lambda) \cap Z(g) \neq \varnothing$ for any g in m_p, from which it follows that $p \in \mathrm{cl}_{\beta_0 X} Z(f-\lambda)$ by [**BB**, Theorem 6]. Thus $Z(\beta_0 f - \lambda) \subset \mathrm{cl}_{\beta_0 X} Z(f-\lambda)$. The reverse inclusion is clear.

(d)$\Rightarrow$(d$'$). It is obvious.

(d$'$)$\Rightarrow$(a). If $p \in \beta_0 X$, then $p \in Z(\beta_0 f - \beta_0 f(p)) = \mathrm{cl}_{\beta_0 X} Z(f - \beta_0 f(p))$ and so $(f - \beta_0 f(p)) \in m_p$.

(b)$\Rightarrow$(e). By the above $f(X)$ is compact. Let $p \in \beta_0 X - \nu_0 X$. Then there is a sequence (V_n) of clopen neighbourhoods of p in $\beta_0 X$ such that $\bigcap V_n \cap X = \varnothing$ and such that $V_n \cap X \supset V_{n+1} \cap X$. Assume there is no neighbourhood of p in $\beta_0 X$ on which $\beta_0 f$ is constant. Hence there is a sequence $(x_k), x_k \in (V_{n_k} - V_{n_{k+1}}) \cap X$ for some increasing sequence (n_k), such that $f(x_k) \neq f(x_j)$ for $k \neq j$. The set $D = \{x_k / k \in N\}$ is a copy of N, C-embedded in X with respect to E, and $f(D)$ is infinite. This is contradictory to the assumption that (b) holds. So there is a neighbourhood of p in $\beta_0 X$ on which $\beta_0 f$ is constant.

(e)$\Rightarrow$(a). By Lemma 1 we may suppose that E is a complete field, and we may also assume that E is infinite, as $C^{\#}(X,E) = C(X,E)$ if E is finite. Now we make the additional assumption that E has nonmeasurable cardinality. By [**BB**, Theorem 15] one has $\nu_0 X = \nu_E X$, that is, $\nu_0 X$ is the set of all maximal ideals of $C(X,E)$ of codimension one. To prove $f \in C^{\#}(X,E)$ it suffices to see that if $p \in \beta_0 X - \nu_0 X$ and $\lambda = \beta_0 f(p)$, then $(f - \lambda) \in m_p$ or, equivalently, that $p \in \mathrm{cl}_{\beta_0 X} Z(f-\lambda)$. This last condition is true as by hypothesis $\beta_0 f$ is constantly equal to λ on a neighbourhood of p in $\beta_0 X$.

The following result due to K. Nowinski [**N**, Theorem 2] will be used in the next corollary:

"If $f\colon Z \to Y$ is a closed continuous map from a metacompact locally compact Hausdorff space Z to a compact space Y, then there exists a compact $K \subset Z$ such that $f(Z-K)$ is finite".

But first we need a lemma:

LEMMA 4. *Let $f \in C^{\#}(X,E)$ and assume X is ultranormal. Then f is a closed map.*

PROOF. Let B be a closed subset of $X, p \in \mathrm{cl}_{\beta_0 X} B$ and $\lambda = \beta_0 f(p)$. Assume $Z(f-\lambda) \cap B = \emptyset$. Since X is ultranormal, there is a clopen subset A of X such that $Z(f-\lambda) \subset A$ and $B \subset X - A$. Let e_A be the E-valued characteristic function of A. Since $Z(e_A) \supset B$, one has $e_A \in m_p$. On the other hand, since $\lambda = \beta_0 f(p)$ and $f \in C^{\#}(X,E)$, one also has $(f-\lambda) \in m_p$. As $Z(f-\lambda) \cap Z(e_A) = \emptyset$ we get a contradiction. So $f(B) = \beta_0 f(\mathrm{cl}_{\beta_0 X} B)$ is a closed subset of E.

COROLLARY 1. *In order that $C^{\#}(X,E) = F(X,E)$ it suffices that any of the following conditions holds*:

(a) *X is an ultranormal metacompact and locally compact space,*

(b) *E is a complete field with nonmeasurable cardinal and X is an E-replete locally compact space,*

(c) *the valuation of E is trivial.*

PROOF. From (a) and (b) of Theorem 1 (cf. [**D$_1$**, Proposition 6]), it is evident that $F(X,E) \subset C^{\#}(X,E)$, so we shall prove the converse. Take $f \in C^{\#}(X,E)$. Assume that (a) holds. From Lemma 4 and Theorem 1, f is a closed map and $f(X)$ is compact, so $f \in F(X,E)$ by the above result of Nowinski. If the valuation of E is trivial then, for $f \in C^{\#}(X,E)$, $f(X)$ is compact and thus finite, and so $C^{\#}(X,E) \subset F(X,E)$. Thus it only remains to prove the inclusion in case E is an infinite field and (b) holds. In this situation one has $X = \nu_E X = \nu_0 X$, and the result follows directly from the equivalence of conditions (a) and (e) in Theorem 1, taking into account the compactness of $\beta_0 X - X$.

REMARK. Note that an ultraregular paracompact locally compact space satisfies condition (a) in Corollary 1 (see [**E**, §1 and **V**, p. 40]). So Corollary 1 strengthens [**D$_2$**, Theorem].

EXAMPLES. Example 1 below shows that no condition (a), (b) or (c) in Corollary 1 is necessary in order to have $C^{\#}(X,E) = F(X,E)$. On the other hand, Examples 2 and 3 prove that neither metacompactness nor local compactness can be dropped in (a).

Let Q_p be the field of the p-adic numbers and $\overline{Q}_p$ the algebraic closure of Q_p. Extend to $\overline{Q}_p$ the p-adic absolute value. Let Ω_p be the completion of $\overline{Q}_p$ and again extend to Ω_p the absolute value on $\overline{Q}_p$. In this way Ω_p is a complete field with respect to a (nonarchimedean) absolute value which extends the p-adic absolute value on Q. Moreover, Ω_p is algebraically closed and therefore it is not locally compact.

For the following examples set $E = \Omega_p$.

EXAMPLE 1 (SEE [**GJ**, p. 123]). Let W be the set of all ordinals less than the first uncountable ordinal endowed with the interval topology. W is an ultranormal locally compact space which is neither metacompact nor E-replete. Nevertheless, $C^{\#}(W,E) = F(W,E) = C(W,E)$.

EXAMPLE 2. Let $X = \Omega_p$. Then X is an ultranormal metacompact E-replete space, but we shall see that $C^{\#}(X,E) \neq F(X,E)$. To see this, set $X_0 = \{\alpha \in X/1 \leq |\alpha|\}$ and $X_n = \{\alpha \in X/1/p^n \leq |\alpha| < 1/p^{n-1}\}$, $n = 1,2,\ldots$. As the sets

X_n ($n = 0, 1, 2, \dots$) are clopen, the function $f\colon X \to E$, given by $f(0) = 0$ and $f(\alpha) = p^n$ for $\alpha \in X_n$, is continuous, and it is clear that $f \notin F(X, E)$. On the other hand, if D is a C-embedded copy of N, then $D - \{0\}$ is bounded away from 0, so $f(D)$ is finite. From (a) and (b) of Theorem 1, it follows that $f \in C^{\#}(X, E)$.

EXAMPLE 3. Let $X = W \times W$. Then X is an ultranormal locally compact space, but $C^{\#}(X, E) = C(X, E) \neq F(X, E)$. (To see that X is an ultranormal space, note that, by Glicksberg's theorem (or [**GJ**, 8 M2]), one has $\beta X = \beta W \times \beta W$. Hence βX is an ultraregular compact space, and so the large inductive dimension of $\beta X, \mathrm{Ind}(\beta X)$, is 0. On the other hand, since X is a normal space, one has $\mathrm{Ind}(X) = \mathrm{Ind}(\beta X)$ (see [**I**, Theorem 8, p. 100]). Thus $\mathrm{Ind}(X) = 0$, and so X is an ultranormal space.)

EXAMPLE 4 (SEE [**N**, EXAMPLE 3 AND $\mathbf{D_3}$, EXAMPLE 3]). Let D be a discrete space of power c, D^* the one-point compactification $D \cup \{w_1\}$ of D, N^* the one-point compactification $N \cup \{w\}$ of N, and $X = N^* \times D^* - \{(w, w_1)\}$. Then X is an ultraregular metacompact space which is neither ultranormal nor E-replete, nevertheless $C^{\#}(X, E) = F(X, E)$.

As usual, $C_K(X, E)$ will denote the ideal of $C(X, E)$ consisting of those functions with compact support. An ideal J of $C(X, E)$ will be called free if

$$\bigcap \{Z(f)/f \in J\} = \varnothing.$$

COROLLARY 2. *Assume that any of the following conditions holds:*

(a) *X is ultranormal metacompact and locally compact,*

(b) *E is a complete field with nonmeasurable cardinal and X is E-replete,*

(c) *the valuation on E is trivial.*

Then $C_K(X, E) = \bigcap \{m/m$ is a free maximal ideal of $C(X, E)\}$.

PROOF. As in [**GJ**, 4D] $C_K(X, E)$ is contained in every free (maximal) ideal of $C(X, E)$. To prove the reverse inclusion take f belonging to every free maximal ideal of $C(X, E)$. It is clear that $f \in C^{\#}(X, E)$. First assume (b). Then from Theorem 1, for any $p \in \beta_0 X - X$ there is a neighbourhood of p in $\beta_0 X$ on which the function $\beta_0 f$ vanishes, so $\beta_0 f$ vanishes on an open neighbourhood of $\beta_0 X - X$, whence $f \in C_K(X, E)$. Now assume (a) or (c). Then from Corollary 1, $f \in F(X, E)$. Let K be a compact subset of X such that $f(X - K) = \{\lambda_1, \dots, \lambda_n\}$. Since the support of f is contained in the set $K \cup \bigcup \{Z(f - \lambda_i),\ 1 \le i \le n, \lambda_i \neq 0\}$, to complete the proof it suffices to see that $Z(f - \lambda_i)$ is compact for $\lambda_i \neq 0$. But this is true because, reasoning as in [**GJ**, p. 58], one deduces that, if $Z(f - \lambda_i)$ were not compact, then $f - \lambda_i$ would belong to some free maximal ideal m of $C(X, E)$, which contradicts the fact $f \in m$.

REMARK. Corollary 2 strengthens [$\mathbf{D_2}$, Corollary].

THEOREM 2. *If either $A = C^{\#}(X, E)$ or $A = F(X, E)$, then $\mathcal{M}(C(X, E)) \to \mathcal{M}(A)$ given by $m \mapsto m \cap A$ is a homeomorphism.*

PROOF. In both cases A is a subalgebra of $C(X, E)$ containing all the idempotents of $C(X, E)$, and A is closed under inversion (i.e., if $f \in A$ and $Z(f) = \varnothing$, then $1/f \in A$). We shall see that the conclusion of the theorem is true for any such algebra.

First we shall show that every maximal ideal of A is of type $m \cap A$ for some maximal ideal m of $C(X, E)$. Let $M \in \mathcal{M}(A)$ and $f_1, \dots, f_n \in M$. If $\bigcap Z(f_i) = \varnothing$, then there are idempotents $e_1, \ldots, e_n$ such that $\sum f_i e_i$ is a unit of $C(X,E)$ (see [**D₁**, Lemma]) and, since A is closed under inversion, then $\sum f_i e_i$ is also a unit in A, which is contradictory to $\sum f_i e_i \in M$. So $\bigcap Z(f_i) \neq \emptyset$ and therefore there is a maximal ideal m of $C(X,E)$ containing M. Then one has $M \subset m \cap A$ and, by the maximal character of M, one concludes that $M = m \cap A$.

Now let $m' \in \mathcal{M}(C(X,E))$. Then $m' \cap A$ is a proper ideal of A, and hence there is $M \in \mathcal{M}(A)$ such that $m' \cap A \subset M$. As we have just seen above, $M = m \cap A$ for some $m \in \mathcal{M}(C(X,E))$, so $m' \cap A \subset m \cap A$. From this inclusion and the assumptions on A, it follows that $m = m'$ (see [**D₁**, Corollary to Proposition 2]). This shows that $m' \cap A$ is a maximal ideal of A. The same argument shows that the map $m \mapsto m \cap A$ is injective and, as for the maximal ideals of $C(X,E)$, two maximal ideals of A containing the same idempotents agree. Hence finally one deduces that the map $m \mapsto m \cap A$ is an homeomorphism since it is a one-to-one and onto continuous map between two compact Hausdorff spaces.

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