PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 3, July 1986

# ON A CONNECTED DENSE PROPER SUBGROUP OF $R^2$ WHOSE COMPLEMENT IS CONNECTED

RYUJI MAEHARA

ABSTRACT. A simple proof is given to the theorem of F. B. Jones which asserts the existence of such a subgroup of the additive group $R^2$ as in the title.

A real-valued function $f\colon R \to R$ is said to be *additive* if it satisfies

$$f(x+y) = f(x) + f(y) \qquad (x, y \in R).$$

If we regard $R$ as a vector space over the rational numbers $Q$, then $f$ is additive if and only if $f$ is $Q$-linear. The graph $G_f = [(x, f(x))\colon x \in R]$ of an additive function $f$ is a subgroup of $R^2$. The purpose of this note is to give a simple proof to the following theorem of F. B. Jones (Theorem 5 and Property 1 in [**1**]).

THEOREM. *There exists an additive function $f\colon R \to R$ such that both the graph $G_f$ and its complement $G_f^c$ are connected and dense in the plane $R^2$.*

Let $p$ denote the perpendicular projection of the plane $R^2$ onto the $x$-axis $R$ $(= R \times 0)$. We say that a set $M$ in $R^2$ has a *positive width* if $p(M)$ contains an open interval of the $x$-axis. We denote by $\Omega$ the collection of all closed subsets of $R^2$ which have a positive width. Then the preceding theorem is a consequence of the following two propositions:

PROPOSITION A. *There exist an additive function $f\colon R \to R$ such that its graph $G_f$ intersects every member $M$ of $Q$.*

PROPOSITION B. *If a graph $G_h$ intersects every member $M$ of $\Omega$, then both $G_h$ and $G_h^c$ are dense connected subsets of $R^2$.*

To prove the propositions, we begin with the following

LEMMA. *$\Omega$ and $R$ have the same cardinality*: $|\Omega| = |R|$.

PROOF. Every straight line parallel to the $x$-axis is a member of $\Omega$. Hence $|R| \leqslant |Q|$. Let $[U_1, U_2, U_3, \ldots]$ be a countable basis for the topology of $R^2$. Every open set $G$ of $R^2$ is the union of all $U_n$ which are contained in $G$. For each $M \in \Omega$, we define

$$s(M) = [n\colon U_n \subset M^c],$$

Received by the editors April 22, 1985 and, in revised form, July 9, 1985.
1980 *Mathematics Subject Classification*. Primary 54G15.

Then $s\colon \Omega \to 2^N$ is injective, where $N$ is the set of all positive integers. Hence, $|\Omega| \leqslant |2^N| = |R|$ and therefore, $|\Omega| = |R|$.

PROOF OF PROPOSITION A. Let $X = [x_j\colon j \in J]$ be a $Q$-basis (i.e. a Hamel basis) of vector space $R$ over $Q$. Then, $|X| \leqslant |R| \leqslant |X| \times |Q| = |X|$, and hence $|X| = |R|$. So there is a one-to-one correspondence $x_j \leftrightarrow M_j$ $(j \in J)$ between $X$ and $\Omega$. For each $j \in J$, since $p(M_j)$ contains an open interval and since $[q \cdot x_j\colon q \in Q]$ is dense in $R$, there exists a $q_j \in Q$ $(q_j \neq 0)$ such that $q_j x_j \in p(M_j)$. Hence, we can find $y_j \in R$ such that $(q_j x_j, y_j) \in M_j$. Let $x_j' = q_j x_j$. Then $X_1 = [x_j'\colon j \in J]$ is also a $Q$-basis. Define $f_1\colon X_1 \to R$ by $f_1(x_j') = y_j$ $(j \in J)$, and extend $f_1$ to a $Q$-linear function $f\colon R \to R$. Then $f$ satisfies the required condition.

PROOF OF PROPOSITION B. Every nonempty open set in the plane contains a member $M$ of $\Omega$, and $M$ contains a point of $G_h$. Hence, $G_h$ is dense. Let $k(x) = h(x) + 1$ $(x \in R)$. Then graph $G_k$ is a translation of $G_h$ and hence dense in $R^2$. Clearly $G_k \subset G_h^c$ and so $G_h^c$ is also dense. By the fact $G_k \subset G_h^c \subset \overline{G}_k$ $(= R^2)$, we know that if $G_k$ is connected then so is $G_h^c$. Since $G_k$ is homeomorphic to $G_h$, the only one we need is to show that $G_h$ is connected.

Suppose that $G_h$ is the sum of mutually separated nonempty sets $H$ and $K$:

$$G_h = H \cup K, \quad \overline{H} \cap K = 0 = H \cap \overline{K}, \quad H \neq 0 \neq K.$$

Let $M = \overline{H} \cap \overline{K}$. Then $M$ separates the plane $R^2$. In fact,

$$\begin{aligned} &R^2 - M = (\overline{H} \cap \overline{K})^c = \overline{H}^c \cup \overline{K}^c, \\ &\overline{H}^c \cap \overline{K}^c = (\overline{H} \cup \overline{K})^c = \overline{G}_h^c = 0, \\ &\overline{H}^c \supseteq K \neq 0, \quad \text{and similarly} \quad \overline{K}^c \neq 0. \end{aligned}$$

$M$ $(\subset \overline{K}, \overline{H})$ intersects neither $H$ nor $K$, and hence $G_h \cap M = 0$. By the assumption, $M$ does not have a positive width. Thus the complement of $p(M)$, say $A$, in the $x$-axis is dense. The open set $R^2 - M$ contains graph $G_h$ and every line $p^{-1}(a)$ for $a \in A$. This is a contradiction to the following lemma:

LEMMA. *Let $A$ be a dense subset of the $x$-axis $R$. Suppose that an open subset $W$ of the plane satisfies the conditions*:

(i) $W \cap p^{-1}(x) \neq 0$ *for all* $x \in R$, *and*

(ii) $p^{-1}(a) \subset W$ *for all* $a \in A$.

*Then $W$ is path-connected.*

PROOF. Let $a_0 \in A$ and let $W_0$ be the path component of $W$ which contains $a_0$. For any $z \in W$, we can find a path-connected neighborhood $U(z)$ of $z$ which is contained in $W$. Since $A$ is dense in the $x$-axis, $U(z)$ meets a line $p^{-1}(a)$ for some $a \in A$. Hence, $z$ and $a$ can be joined by a path in the subset $U(z) \cup p^{-1}(a)$ of $W$.

Now we prove that $a \in W_0$. Suppose $a \in W - W_0$. We may assume $a_0 < a$. Let $A_1 = [b \in A\colon a_0 < b,\ b \in W - W_0]$, and let $\xi = \inf A_1$. Since $W_0$ is an open set containing $a_0$, we have $a_0 < \xi$. By (i), there is a point $z_1 \in W \cap p^{-1}(\xi)$. Let $U(z_1)$ be a path-connected neighborhood of $z_1$ contained in $W$. Then $p(U(z_1))$ is a neighborhood (in the $x$-axis) of $\xi$, and contains a point $b' \in A_1$ and a point $a' \in A$ such that $a_0 \leqslant a' < \xi \leqslant b'$. By the definition of $\xi$, $a' \in W_0$. Points $a'$ and $b'$ are in $p^{-1}(a') \cup U(z_1) \cup p^{-1}(b')$ which is a path-connected subset of $W$. Hence, $b' \in W_0$. This contradicts that $b' \in A_1$, and completes the proof.

## References

1. F. B. Jones, *Connected and disconnected plane sets and the functional equation $f(x) + f(y) = f(x + y)$*, Bull. Amer. Math. Soc. **48** (1942), 115–120.

Department of Mathematics, University of Ryukyu, Nishihara, Okinawa 903 - 01, Japan