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CHARACTER VALUES OF FINITE GROUPS AS EIGENVALUES OF NONNEGATIVE INTEGER MATRICES

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ABSTRACT. Let $C_1, C_2, \ldots, C_k$ be the conjugacy classes of the finite group G and choose $x_i \in C_i$, for $i = 1, 2, \ldots, k$. For every complex character θ of G there is a $k \times k$ matrix $M(\theta)$ whose entries are nonnegative integers such that $X^{-1}M(\theta)X = \operatorname{diag}(\theta(x_1), \theta(x_2), \ldots, \theta(x_k))$ where X is the character table matrix of G. Some consequences are shown.

For a finite group G we fix the following notation. Let $C_1, C_2, \ldots, C_k$ be all the conjugacy classes of G and $\operatorname{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_k\}$ the set of irreducible complex characters of G. We choose a representative $x_i \in C_i$ for each $i = 1, 2, \ldots, k$. Denote by X the character table matrix of G, that is: $X = (\chi_i(x_j))$.

If θ is any complex character of G we define $M(\theta)$ to be the $k \times k$ matrix, $M(\theta) = (m_{ij}(\theta))$, where $m_{ij}(\theta) = [\theta\chi_i, \chi_j]$ for all $i, j = 1, 2, \ldots, k$. The characteristic polynomial of $M(\theta)$ will be denoted by $p_\theta(x)$ and the minimal polynomial of $M(\theta)$ will be denoted by $m_\theta(x)$. The matrices $M(\theta)$ were used, for example, in Lemma (10.4), Chapter IV of [**4**]. In fact, our Theorem A below is similar to that lemma.

Let Z be the ring of integers, Q and $\mathbf{C}$ the rational and complex fields respectively and $Z[X]$, $\mathbf{C}[x]$ the corresponding polynomial rings.

We remark that $M(\theta)$ is a matrix whose entries are all nonnegative integers (that is, a nonnegative integral matrix). Thus $p_\theta(x) \in Z[x]$. As $m_\theta(x)$ divides $p_\theta(x)$ in $Q[x]$ we have that $m_\theta(x) \in Q[x]$. From linear algebra we know that $m_\theta(x)$ is the minimal polynomial of $M(\theta)$ even if $M(\theta)$ is regarded as a matrix over $\mathbf{C}$. Finally, we denote by $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ the diagonal $n \times n$ matrix whose main diagonal's elements are $a_1, a_2, \ldots, a_n$.

THEOREM A. *Let θ be any complex character of the finite group G and let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be all the distinct values taken on by θ. If $X, M(\theta), p_\theta(x), m_\theta(x), \chi_i, x_i$, $i = 1, 2, \ldots, k$, are as defined above, then*

(a) $p_\theta(x) = \prod_{i=1}^{k}(x - \theta(x_i)) \in Z[x]$.
(b) $m_\theta(x) = \prod_{i=1}^{m}(x - \alpha_i) \in Z[x]$.
(c) $X^{-1} \cdot M(\theta) \cdot X = \operatorname{diag}(\theta(x_1), \theta(x_2), \ldots, \theta(x_k))$.

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In particular the values $\theta(x_1)$, $\theta(x_2), \dots, \theta(x_k)$, of θ, are the eigenvalues of the diagonalizable nonnegative integral matrix $M(\theta)$.

The proof is presented at the end of the article. It is a straightforward short computation and uses nothing but orthogonality relations and elementary linear algebra. We note that the fact that the polynomials $\prod_{i=1}^{k}(x - \theta(x_i))$ and $\prod_{i=1}^{m}(x - \alpha_i)$ have integer coefficients can be shown also by using Galois theory. Here it will follow from the definition of $p_\theta(x)$ and $m_\theta(x)$. Some known results on characters of finite groups are consequences of Theorem A. We list them next.

1. The fact that character values are algebraic integers is demonstrated directly without using the fact that all algebraic integers form a ring.

2. The character table of a group G can be computed from the character algebra structure constants g_{ijr} which are defined as follows.

$$\chi_i \chi_j = \sum_{r=1}^{k} g_{ijr} \chi_r.$$

3. Using the notation of Theorem A, assume that $\alpha_1 = \theta(1)$ and let $f(x) = \prod_{i=2}^{m}(x - \alpha_i)$. By Theorem A, $f(x) \in Q[x]$ and as its coefficients are algebraic integers we get that $f(x) \in Z[x]$. Write $f(x) = \sum_{i=0}^{m-1} a_i x^i$ where the a_i are integers. Then

$$\sum_{g \in G} f(\theta(g)) = \sum_{g \in G} \sum_{i=0}^{m-1} a_i (\theta(g))^i = |G| \sum_{i=0}^{m-1} a_i [\theta^i, 1_G] \equiv 0 \pmod{|G|}.$$

However,

$$\sum_{g \in G} f(\theta(g)) = |\mathrm{Ker}\,\theta| \prod_{i=2}^{m} (\alpha_1 - \alpha_i).$$

As a corollary we get a Galois-theory-free proof of a result which was proved by Blichfeldt [**2**] and Kiyota [**5**] in the case that θ is a permutation character.

COROLLARY. *Let θ be a complex character of degree d of a finite group G. If $\alpha_2, \alpha_3, \dots, \alpha_m$ are all the distinct values of θ which are different from d, then* $|\mathrm{Ker}\,\theta| \cdot \prod_{i=2}^{m}(d - \alpha_i) \equiv 0 \pmod{|G|}$.

We note that the proof of [**2**] is good for the general case as well.

4. Integrality results can be concluded from Theorem A without using the usual Galois theory methods. For example, $\sum_{i=1}^{k} \theta(x_i)$, $\sum_{i=1}^{m} \alpha_i, \prod_{i=1}^{k} \theta(x_i)$, $\prod_{i=1}^{m} \alpha_i$ and in fact every elementary symmetric function in the $\theta(x_i)$'s (respectively, the α_i's) is an integer. Also, every elementary symmetric function (e.g., sum and product) of those $\theta(x_i)$ (respectively, those α_i) which are nonrational is an integer.

Different applications of the matrices $M(\theta)$ can be found in §II [**1**] and §II [**3**].

PROOF OF THEOREM A. Let V be the hermitian vector space of all the complex class functions of G. Define a linear transformation $T(\theta): V \to V$ by $(T(\theta)f)(g) = \overline{\theta(g)} f(g)$. Then $(T^*(\theta)f)(g) = \theta(g)f(g)$ for all $g \in G$. Here $T^*(\theta)$ is the adjoint operator defined by $[v, T(\theta)u] = [T^*(\theta)v, u]$ for all $v, u \in V$.

It follows that $M(\theta)$ is the matrix of $T(\theta)$ with respect to the orthonormal basis, $\mathrm{Irr}(G)$, of V.

Let $f_r \in V$ be defined by $f_r(x_j) = \delta_{jr} \cdot |G|/|C_r|$ for $r = 1, 2, \ldots, k$. For each index r, $1 \le r \le k$, define r' such that $x_r^{-1} = x_{r'}$. Clearly $|C_r| = |C_{r'}|$. Then the set $B = \{f_{r'} | r = 1, 2, \ldots, k\}$ is another basis of V and as $T(\theta) f_{j'} = \theta(x_j) f_{j'}$ for each j', we get that the matrix of $T(\theta)$ with respect to B is $\operatorname{diag}(\theta(x_1), \theta(x_2), \ldots, \theta(x_k))$. Now the orthogonality relation $f_{j'} = \sum_{i=1}^{k} \chi_i(x_j)\chi_i$ implies that the transition matrix from the basis $\operatorname{Irr}(G)$ to the basis B is X. Now parts (a) and (c) follow. To get (b) we recall that the minimal polynomial of a diagonalizable matrix over $\mathbf{C}$ is the product of all the distinct linear factors of the characteristic polynomial. Hence $\prod_{i=1}^{m}(x - \alpha_i) = m_\theta(x) \in Q[x]$. But then the coefficients of $m_\theta(x)$ are algebraic integers, and therefore they are integers.

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