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# FINITE RANK PERTURBATIONS OF SINGULAR SPECTRA

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ABSTRACT. Let $T$ be selfadjoint, and $V$ nonnegative of finite rank, with the range of $V$ cyclic for $T$. Then the singular parts of $T$ and $H = T + V$ are supported on two sets $S_T$ and $S_H$ such that the multiplicity of $T$ on $S_T \cap S_H$ is less than the rank of $V$.

**1. Introduction.** In [**3**], Donoghue, following earlier work of Aronszajn, proved

1. THEOREM. *Let $\phi$ be a cyclic vector for a selfadjoint operator $T$. For real $c \neq 0$, the singular parts of $T$ and $H = T + c\langle\cdot,\phi\rangle\phi$ are supported on disjoint sets.*

To generalize this result to perturbations of rank higher than one is not completely straightforward, as a consideration of matrix examples will easily show. A certain generalization to nonnegative perturbations was given by the author in [**4**]. The criterion of that paper will be applied here to prove the following result:

2. THEOREM. *Let $T$ be selfadjoint, $V$ a nonnegative operator of finite rank, and $H = T + V$. Assume that the range of $V$ is cyclie for $T$. Let $\mu_T$ and $n_T(\lambda)$ be a scalar spectral measure and multiplicity function of $T$, and define*

$$G = \{\lambda : n_T(\lambda) = \operatorname{rank} V\}.$$

*Then there exist sets $S_T$ and $S_H$ supporting the singular parts of $T$ and $H$ such that $S_T \cap G$ and $S_H$ are disjoint.*

Note that $n_T(\lambda)$ cannot exceed the rank of $V$ when the range of $V$ is cyclic. This has the following corollary, which is interesting even for matrices.

3. COROLLARY. *Let $T$ be selfadjoint, $V$ nonnegative of finite rank, and the range of $V$ cyclic for $T$. If $\lambda$ is an eigenvalue of $T$ with multiplicity equal to the rank of $V$, then $\lambda$ is not an eigenvalue of $H = T + V$.*

For related work, see also [**1, 2, 5**].

**2. Proofs.** Note that the basic Hilbert space $\mathcal{H}$ is separable because the range $V\mathcal{H}$ of $V$ is cyclic. Multiplicity theory is therefore applicable.

For $\varepsilon > 0$, define the function

$$\delta_\varepsilon(t) = \frac{1}{\pi}\frac{\varepsilon}{t^2 + \varepsilon^2}.$$

As observed by Donoghue [**3**, §1], the singular part of $T$ is supported by the measurable set

$$S_T = \left\{\lambda : \lim_{\varepsilon\downarrow 0}\langle\delta_\varepsilon(T-\lambda)x, x\rangle = \infty \text{ for some } x \in V\mathcal{H}\right\}$$

and similarly for $H$.

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Let $v_1 \geq v_2 \geq \cdots \geq v_r > 0$ be the nonzero eignevalues of $V$, and $\phi_1, \ldots, \phi_r$ the corresponding normalized eigenvectors, which therefore are an orthonormal basis of $V\mathcal{H}$. Because $V\mathcal{H}$ is cyclic, one may choose

$$\mu_T(S) = \sum_{j=1}^{r} \langle E_T[S]\phi_j, \phi_j \rangle$$

as the scalar spectral measure, where $E_T$ is the spectral measure of $T$. Thus, $\langle E_T(d\lambda)\phi_i, \phi_j \rangle$ is absolutely continuous with respect to $\mu_T$. Define $M_{ij}(\lambda)$ $\mu_T$-a.e. to be the Radon-Nikodým derivative

$$\langle E_T(d\lambda)\phi_i, \phi_j \rangle = M_{ij}(\lambda)\mu_T(d\lambda)$$

and $M(\lambda)$ to be the nonnegative matrix $M(\lambda) = \{M_{ij}(\lambda)\}_{i,j=1,\ldots,r}$. We shall regard $M(\lambda)$ as an operator on the space $V\mathcal{H}$. Let $m(\lambda)$ be the smallest eigenvalue of $M(\lambda)$:

$$m(\lambda) = \inf\{\langle M(\lambda)u, u \rangle : |u|^2 = 1, \ u \in V\mathcal{H}\},$$

where $|u|$ is the norm of $u$ in $\mathcal{H}$. Since $u$ may be restricted to a countable dense set, $m(\lambda)$ is measurable. One has

$$M(\lambda) \geq m(\lambda)P \tag{1}$$

where $P$ is the projection onto $V\mathcal{H}$. Note also that

$$m(\lambda) \leq 1. \tag{2}$$

Clearly, for all Borel sets $S$, $\langle E_T[S]\phi_i, \phi_i \rangle \leq \mu_T(S)$ so that $M_{ii}(\lambda) \leq 1$ $\mu_T$-a.e., and hence

$$m(\lambda) \leq \min_{1 \leq i \leq r} M_{ii}(\lambda) \leq 1.$$

4. LEMMA. *One has*

$$n_T(\lambda) = \operatorname{rank} M(\lambda) \quad \mu_T\text{-a.e.}$$

PROOF. This undoubtedly follows from the readers' favorite version of multiplicity theory. The *author's* favorite version is the Kato-Kuroda construction of direct integrals by spectral forms [**6**, **7**]. In that terminology, let $\mathcal{X} = V\mathcal{H}$, and

$$f(\lambda, u) = \sum_{ij=1}^{r} u_i \overline{u}_j M_{ij}(\lambda) \equiv \langle M(\lambda)u, u \rangle$$

for $u = u_1\phi_1 + \cdots + u_r\phi_r \in \mathcal{X}$. Then $(f, \mathcal{X})$ is a spectral form for $T$ with respect to $\mu_T$, and the direct integral

$$\mathcal{H} \cong \int_{\sigma(T)}^{\oplus} \mathcal{X}(\lambda)\mu_T(d\lambda)$$

diagonalizes $T$, where $\mathcal{X}(\lambda)$ is the (completion of) the quotient space $\mathcal{X}/\{u \in \mathcal{X} : f(\lambda, u) = 0\}$. In this case, no completion is needed, since $\mathcal{X}(\lambda)$ is the finite-dimensional space $V\mathcal{H}/\ker M(\lambda)$ whose dimension is $\operatorname{rank} M(\lambda)$. □

The theorem of [**4**] will now be recalled. Let $\mathcal{K}$ be another Hilbert space, and $A: \mathcal{K} \to \mathcal{H}$ bounded. Let $T$ be selfadjoint, and assume that $A\mathcal{K}$ is cyclic for $T$.

5. PROPOSITION. *The singular part of* $H = T + AA^*$ *is supported on the complement of the set of points* $\lambda$ *for which there is an* $\eta > 0$ *such that*

$$A^*\delta_\varepsilon(T-\lambda)A \geq \eta I \tag{3}$$

*for all sufficiently small* $\varepsilon > 0$.

Note that $I$ in (3) is the *identity on* $\mathcal{K}$, not $\mathcal{H}$. Note also that Proposition 5 implies Theorem 1 if one takes $\mathcal{K} = \mathbf{C}$ (the complex numbers) and $A^* = c^{1/2}\langle\cdot,\phi\rangle$ (not $A$, as the misprint in [**4**] has it). Note finally that although $T$ was assumed bounded in [**4**], the proof there goes through *unchanged* for unbounded $T$.

To prove Theorem 2, factor $V = AA^*$ through the space $\mathcal{K} = V\mathcal{H} = V^{1/2}\mathcal{H}$ by defining $A: V^{1/2}\mathcal{H} \to \mathcal{H}$ as $Au = V^{1/2}u$. Then $A^*: \mathcal{H} \to V^{1/2}\mathcal{H}$ is also $A^*u = V^{1/2}u$, and $V = AA^*$. The identity $I$ in (3) is now the projection $P$ onto $V\mathcal{H}$.

For fixed $\lambda$ and $u \in V\mathcal{H}$, one has

$$\begin{aligned}\langle A^*\delta_\varepsilon(T-\lambda)Au, u\rangle &= \langle \delta_\varepsilon(T-\lambda)V^{1/2}u, V^{1/2}u\rangle \\ &= \sum_{i,j=1}^{r} u_i\overline{u}_j v_i^{1/2}v_j^{1/2}\langle\delta_\varepsilon(T-\lambda)\phi_i,\phi_j\rangle \\ &\geq v_r\langle\delta_\varepsilon(T-\lambda)u,u\rangle = v_r\int\delta_\varepsilon(t-\lambda)\langle M(t)u,u\rangle\mu_T(dt) \\ &\geq v_r\|u\|^2\int\delta_\varepsilon(t-\lambda)m(t)\mu_T(dt)\end{aligned}$$

where (1) was used at the last step. Let $F$ be the set of all $\lambda$ for which

$$\lim_{\varepsilon\downarrow 0}\int\delta_\varepsilon(t-\lambda)m(t)\mu_T(dt) = \infty.$$

By (2), $F \subset S_T$, while by Proposition 5, its complement $F^c$ supports the singular part of $H$. (In fact, by the proof in [**5**], $S_H \subset F^c$.)

Now [**3**, §1] $F$ supports the singular part of the measure $m(t)\mu_T(dt)$. The measure $\chi_G(t)\mu_T(dt)$ has the same null sets, because $G = \{t: m(t) > 0\}$, and its singular part is supported by $S_T \cap G$. Thus $F$ and $S_T \cap G$ differ only by a set of $\mu_T$-measure zero. The result is then obtained by replacing $S_T$ by $S_T' = S_T \sim (G \cap S_T \cap F^c)$. □

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