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A FIXED POINT THEOREM FOR LOCALLY NONEXPANSIVE MAPPINGS IN NORMED SPACES

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ABSTRACT. It is shown that global conditions in a recent result of W. A. Kirk can be replaced with the corresponding local conditions in case the domain is connected. Also a remark is made about the proof of the theorem referenced.

1. In this paper we adopt the notation of [2]. Let X be a compact subset of a normed linear space E, and $T\colon X \to X$ be a mapping. We let $\Delta' X$ denote the boundary of X in $\overline{\mathrm{co}}\, X$. The mapping T is locally nonexpansive (contractive) if for each $x \in X$ there exists $\varepsilon > 0$ so that whenever y and z are distinct points in X and $y, z \in B(x, \varepsilon)$, $\|T(y) - T(z)\| \leqslant \|y - z\|$ ($\|T(y) - T(z)\| < \|y - z\|$). The mapping T is called nonexpansive (contractive) if for each $x \in X$, ε is unbounded.

A metric space (X, d) is chainable if for each $\varepsilon > 0$ and points x and y in X, there exists a finite set of distinct points $x = x_1, \ldots, x_n = y$ in X so that $d(x_i, x_{i+1}) \leqslant \varepsilon$ for each $i = 1, \ldots, n - 1$.

Rosenholtz [3] proved the following lemma.

LEMMA 1. *Let (X, d) be a compact and connected metric space. Then for each $\varepsilon > 0$ and $x, y \in X$ there exists an ε-chain between x and y, and the mapping $d_\varepsilon\colon X \times X \to R$, defined by*

$$(1) \quad d_\varepsilon(x, y) = \inf\left\{ \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \,\middle|\, x = x_1, \ldots, x_n = y \text{ is an } \varepsilon\text{-chain between } x \text{ and } y \right\},$$

is a metric on X equivalent to d. Furthermore, for each $x, y \in X$ and $\varepsilon > 0$ there exists an ε-chain $x = x_1, \ldots, x_n = y$ so that

$$(2) \qquad d_\varepsilon(x, y) = \sum_{i=1}^{n-1} d(x_i, x_{i+1}).$$

2.

LEMMA 2. *Let X be a compact connected subset of a Banach space. If $f\colon X \to X$ is locally nonexpansive on X and locally contractive on $\Delta' X$ there exists $\delta > 0$ so that f is*

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nonexpansive and locally contractive on $\Delta' X$ with respect to d_δ as defined in (1), *and* $d_\delta(x, y) = \|x - y\|$ *if either* $\|x - y\| \leq \delta$ *or* $[x, y] \subseteq X$.

PROOF. By the compactness of X there exists $\delta_1 > 0$ so that if $x, y \in X$ and $\|x - y\| < \delta_1$, then $\|f(x) - f(y)\| \leq \|x - y\|$. Also by the compactness of $\Delta' X$ there exists $\delta_2 > 0$ so that for all distinct points $x, y \in \Delta' X$ with $\|x - y\| < \delta_2$, $\|f(x) - f(y)\| < \|x - y\|$. Let $\delta = 2^{-1}\min\{\delta_1, \delta_2\}$. By Lemma 1 we may choose the metric $d \equiv d_\delta$ as defined in (1) to remetrize X. The second assertion in Lemma 2 easily follows from the definition of d and the triangle inequality of the metric induced by the norm. To see the first assertion, let $x, y \in X$ and $x = x_1, \dots, x_n = y$ be a δ-chain in X from x to y satisfying (2).

By the local nonexpansiveness of f

$$(3) \qquad d(f(x), f(y)) \leq \sum_{i=1}^{n-1} d(f(x_i), f(x_{i+1})) = \sum_{i=1}^{n-1} \|f(x_i) - f(x_{i+1})\| \leq \sum_{i=1}^{n-1} \|x_i - x_{i+1}\| = d(x, y).$$

Therefore f is nonexpansive with respect to d.

The local contractiveness of f with respect to d follows from the definition of δ, and the fact that $d(x, y) = \|x - y\|$ if $\|x - y\| < \delta$.

LEMMA 3. *Let X, f, satisfy the hypotheses of Lemma* 2, *and d satisfy the conclusion of Lemma* 2. *If $x, y \in X$, $x = x_1, \dots, x_n = y$ is a δ-chain from x to y satisfying* (2), *and $d(f(x), f(y)) = d(x, y)$, then there does not exist consecutive points of $x = x_1, \dots, x_n = y$ in $\Delta' X$.*

PROOF. Suppose there exists $j \in \{1, \dots, n - 1\}$ so that $x_j, x_{j+1} \in \Delta' X$. Then by Lemma 2 and (2),

$$(4) \qquad d(f(x), f(y)) \leq \sum_{i=1}^{n-1} d(f(x_i), f(x_{i+1})) < \sum_{i=1}^{n-1} d(x_i, x_{i+1}) = d(x, y).$$

But (4) contradicts $d(f(x), f(y)) = d(x, y)$.

THEOREM 1. *Let X be a compact nonempty connected subset of a Banach space. If f: $X \to X$ is locally nonexpansive on X and locally contractive on $\Delta' X$, then f has a fixed point.*

PROOF. Let d be the metric as guaranteed in Lemma 2. For a subset A of X, let $\bar{\delta}(A)$ and $\delta(A)$ be the d-diameter and norm diameter of A respectively. By **[2]** we can choose a minimal invariant nonempty subset M of X with minimal d-diameter. By **[1]** f restricted to M is a d-isometry.

We first show $\delta(M) = \bar{\delta}(M)$. By Lemma 2 it is sufficient to show for all $m_1, m_2 \in M$, $[m_1, m_2] \subseteq X$. For $m_1, m_2 \in M$ let $m_1 = x_1, \dots, x_n = m_2$ be a δ-chain satisfying (2). If for some $i = 1, \dots, n - 1$, $[x_i, x_{i+1}] \not\subseteq X$, we may choose distinct points $z_1, z_2 \in [x_i, x_{i+1}] \cap \Delta' X$ so that $\|x_i - x_{i+1}\| = \|x_i - z_1\| + \|z_1 - z_2\| + \|z_2 - x_{i+1}\|$. Then by (1)

$$d(f(m_1), f(m_2)) \leqslant d(f(x_1), f(x_2)) + \cdots + d(f(x_1), f(z_1)) + d(f(z_1), f(z_2)) + \cdots + d(f(x_{n-1}), f(x_n)) < d(m_1, m_2). \tag{5}$$

But (5) contradicts that f is a d-isometry on M. Hence for $i = 1, \ldots, n-1$, $[x_i, x_{i+1}] \subseteq X$.

Let $\sigma \equiv d(m_1, m_2)$, and let g: $[0, \sigma] \to X$ be the arc length parametrization of the polygonal path

$$P \equiv \bigcup_{i=1}^{n-1} [x_i, x_{i+1}], \tag{6}$$

with $g(0) = x_1$ and $g(\sigma) = x_n$. Let

$$A \equiv \{ t \in [0, \sigma] | [m_1, g(s)] \subseteq X \text{ for } s \in [0, t] \}. \tag{7}$$

The set A is trivially nonempty. Let $u \equiv \sup\{t | t \in A\}$. We show $u = \sigma$. Let $\{t_n\}$ be a sequence in A with $\{t_n\} \uparrow u$, and $s \in [0, 1]$. Then,

$$\|(1-s)x_1 + sg(u) - (1-s)x_1 - sg(t_n)\| = s\|g(u) - g(t_n)\|. \tag{8}$$

Since X is closed and g is continuous, (8) implies $(1-s)x_1 + sg(u) \in X$. Hence $[x_1, g(u)] \subseteq X$.

If $u < \sigma$, we can choose a sequence $\{t_n\}$ in $[0, \sigma]$ decreasing to u, and points z_n, y_n in $[m_1, g(t_n)]$ so that $\|z_n - y_n\| > \delta$ and $[z_n, y_n] \subset \operatorname{co} X \setminus X$. If for some n y_n, z_n cannot be chosen so that $\|z_n - y_n\| > \delta$, we can form a δ-chain $m_1 = y_1, \ldots, y_k = g(t_n)$ along $[m_1, g(t_n)]$ containing two consecutive boundary points so that

$$\sum_{i=1}^{k-1} \|y_i - y_{i+1}\| = d(m_1, g(t_n)).$$

Then by (1) $m_1 = y_1, \ldots, y_k, \ x_{j+1}, \ldots, x_n = m_2$, where $g(t_n) \in [x_j, x_{j+1}]$ is a δ-chain satisfying (2) with respect to m_1 and m_2. Applying Lemma 3 we reach a contradiction. By the compactness of $\overline{\operatorname{co}} X$ we may assume there exist distinct points y, z in $\overline{\operatorname{co}} X$ so that $z_n \to z$ and $y_n \to y$. By the continuity of g and the definition of $\Delta' X$

$$[z, y] \subseteq [x_1, g(u)] \cap \Delta' X. \tag{9}$$

Let $g(u) \in [x_j, x_{j+1}]$. Then by (1)

$$d(m_1, m_2) = \|m_1 - g(u)\| + d(g(u), x_{j+1}) + \cdots + d(x_{n-1}, x_n). \tag{10}$$

By (9) we may choose a δ-chain $m_1 = y_1, \ldots, y_k = g(u)$ in $[m_1, g(u)]$ containing two consecutive points in $\Delta' X$ satisfying $\|m_1 - g(u)\| = \sum_{i=1}^{k-1} d(y_i, y_{i+1})$. Then by (10)

$$d(f(m_1), f(m_2)) \leqslant d(f(y_1), f(y_2)) + \cdots + d(f(y_{k-1}), f(y_k)) + d(f(y_k), f(x_{j+1})) + \cdots + d(f(x_{n-1}), f(x_n)) < d(m_1, m_2). \tag{11}$$

But (11) contradicts that f restricted to M is a d-isometry. Thus $[m_1, m_2] \subseteq X$. It now follows that $\delta(M) = \bar{\delta}(M)$.

Note that for each pair of points m_1, m_2 in M with $d(m_1, m_2) \geqslant \delta$ there exists $\varepsilon > 0$ so that if $y, z \in X$ with $y \in B_d(m_1, \varepsilon)$ and $z \in B_d(m_2, \varepsilon)$, then $d(y, z) = \|y - z\|$. If this is not the case, there exist $m_1, m_2 \in M$ with $d(m_1, m_2) \geqslant \delta$; so for each $\varepsilon > 0$ there exist $y_\varepsilon \in B_d(m_1, \varepsilon)$, $z_\varepsilon \in B_d(m_2, \varepsilon)$ and $a_\varepsilon, b_\varepsilon \in [y_\varepsilon, z_\varepsilon] \cap \operatorname{co} X \setminus X$ so that $\|a_\varepsilon - b_\varepsilon\| > \delta$. By the compactness of $\overline{\operatorname{co}} X$ we can choose distinct points a, b in X so that $[a, b] \subseteq [m_1, m_2] \cap \Delta' X$. But then we can form a δ-chain from m_1 to m_2 along $[m_1, m_2]$ containing consecutive points in $\Delta' X$ and apply Lemma 3 to reach a contradiction.

Let $A \equiv \{(u, v) \in M \times M \mid d(u, v) \geqslant \delta\}$. By the compactness of A we can choose a number ε in $(0, 3^{-1}\delta)$ so that if (u, v) is in A and $y, z \in X$ with $y \in B_d(u, \varepsilon)$ and $z \in B_d(v, \varepsilon)$, then $d(y, z) = \|y - z\|$. By the definition of ε and by the triangle inequality,

$$\text{for all } u, v \in M, \text{ and } y \in B_d(u, \varepsilon) \text{ and } z \in B_d(v, \varepsilon),\quad d(y, z) = \|y - z\|. \tag{12}$$

If $\delta(M) \neq 0$, we can choose distinct points $m_1, m_2 \in M$ and $t \in (0, 1)$ so that $x_0 \equiv (1 - t)m_1 + tm_2$ is in the interior of X with respect to $\operatorname{co} X$ and $d(x_0, m_1) < \varepsilon$. Otherwise we can choose a δ-chain satisfying (2) between m_1 and m_2 along $[m_1, m_2]$, containing two consecutive points in $\Delta' X \cap B_b(m_1, \varepsilon)$ and contradict $d(f(m_1), f(m_2)) = d(m_1, m_2)$.

Let N denote the set of nonnegative integers. Also for each $x \in X$, let $w(x) \equiv \overline{\{f^n(x) \mid n \in N\}}$. By the minimality of M, $w(m_1) = M$ and for each $n \in N$, $f^n(M) = M$.

By the normal structure of $\operatorname{co} M$ there exist $y \in \operatorname{co} M$, and a real number r satisfying $0 < r < \delta(M)$ so that $\|y - m\| \leqslant r$ for all $m \in M$.

Since x_0 is in the interior of X with respect to $\operatorname{co} X$ and $y \in \operatorname{co} X$, there exists $s \in (0, 1)$ so that $z \equiv (1 - s)x_0 + sy \in B_d(m_1, \varepsilon) \cap X$. Then for all $m \in M$,

$$\|z - m\| \leqslant (1 - s)\|x_0 - m\| + s\|y - m\| \leqslant (1 - s)\delta(M) + sr. \tag{13}$$

Let $\bar{r} \equiv (1 - s)\delta(M) + sr$. Since $s \in (0, 1)$, $\bar{r} < \delta(M)$. Let $m, m_n \in M$ so that for all $n \in N$, $f^n(m_n) = m$. Then by the nonexpansiveness of f, and (12) and (13)

$$\begin{aligned} d(f^n(z), m) &= d(f^n(z), f^n(m_n)) \leqslant d(z, m_n) \\ &= \|z - m_n\| \leqslant \bar{r}. \end{aligned} \tag{14}$$

We show next that $\bar{\delta}(w(z)) < \bar{\delta}(M)$, which will contradict the definition of M and imply $\bar{\delta}(M) = 0$.

By the continuity of f it suffices to show for all $m, n \in N$, $d(f^n(z), f^m(z)) \leqslant \bar{r}$. Since $z \in \operatorname{co} M$, by (14), $\|f^n(z) - z\| \leqslant \bar{r}$. By (12) for $m, n \in N$ with $m > n$,

$$d(f^m(z), f^n(z)) \leqslant d(f^{m-n}(z), z) = \|f^{m-n}(z) - z\| \leqslant \bar{r}. \tag{15}$$

Therefore $\bar{\delta}(w(z)) \leqslant \bar{r}$. But (15) contradicts the definition of M. Hence $\bar{\delta}(M) = 0$. Since $M \neq \varnothing$, $M = \{m\}$ for some $m \in X$. Therefore $f(m) = m$.

3. We now comment on Theorem (1) in [**2**]. After choosing a minimal nonempty compact invariant set M with minimal diameter, Kirk claimed $\operatorname{co} M \subseteq X$ by showing for all $m_1, m_2 \in M$, $[m_1, m_2] \subseteq X$. Clearly it is not enough to show $\operatorname{co} M \subseteq X$ unless one also shows $[m_1, m_2] \subseteq M$. However, one can avoid this situation by choosing an interior point x_0 as in Theorem 1 and then show $\delta(w(w_0)) < \delta(M)$.

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