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SOME RINGS OF DIFFERENTIAL OPERATORS WHICH ARE MORITA EQUIVALENT TO THE WEYL ALGEBRA A_1

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ABSTRACT. For a certain class of semigroup algebras $k\Lambda$, we show that the ring of all differential operators on $k\Lambda$ is Morita equivalent to the first Weyl algebra A_1.

Let k be a field, and K a commutative k-algebra. We denote by $D_0(K, K)$ the set of K-linear maps $K \to K$, and if $p \geqslant 1$ then a k-linear map $f\colon K \to K$ belongs to $D_p(K, K)$ provided the map $[f, r]$ defined by $[f, r](s) = f(rs) - rf(s)$ for $s \in K$ belongs to $D_{p-1}(K, K)$ for all $r \in K$. The set $D(K) = \bigcup_{p \geqslant 0} D_p(K, K)$ forms a subring of $\mathrm{End}_k(K)$ called the *ring of differential operators on* K. We consider K as a left $D(K)$-module where $f \cdot r = f(r)$ for all $f \in D(K)$, $r \in K$.

In this note we show that for certain subalgebras K of $k[t]$, where k is a field of characteristic zero and t an indeterminate, the ring $D(K)$ is Morita equivalent to the Weyl algebra A_1. If Λ is a subsemigroup of the set of nonnegative integers $\mathbf{N}$, we let $k\Lambda$ be the semigroup algebra on Λ, which we regard as the subalgebra of $k[t]$ spanned by $\{t^\lambda \mid \lambda \in \Lambda\}$.

THEOREM. *If the semigroup algebra $k\Lambda$ has the same quotient field as $k[t]$, then $D(K)$ is Morita equivalent to A_1.*

The present proof is due to S. P. Smith.

The first result is well known, see [**2**, Lemma 2].

LEMMA 1. *If K is a k-algebra and C a multiplicatively closed subset of K consisting of regular elements, then any differential operator on K has a unique extension to K_C. Hence $D(K)$ may be regarded as a subalgebra of $D(K_C)$. In addition,*

$$D(K) = \{x \in D(K_C) \mid x \cdot K \subseteq K\}.$$

We let $A_1 \doteq k[q, p]$ be the k-algebra generated by q and p subject to the relation $pq - qp = 1$, and regard A_1 as $D(k[t])$ where q acts as a multiplication by t and p as d/dt. If K is a subalgebra of $k[t]$ with the same quotient field $k(t)$, then by Lemma 1, we can view both $D(K)$ and A_1 as subalgebras of the quotient ring Q of A_1. We set $I = \{x \in A_1 \mid x \cdot k[t] \subseteq K\}$ and $S = \{x \in Q \mid xI \subseteq I\}$. Then $S \cong \mathrm{End}_{A_1}(I)$, see [**1**, p. 69].

LEMMA 2. $D(K) \subseteq S$.

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PROOF. If $x \in D(K)$, then x acts on $k(t)$. Also $(xI)k[t] = x(Ik[t]) \subseteq xK \subseteq K$. Therefore, $xI \subseteq I$ and so $x \in S$.

PROOF OF THE THEOREM. We now assume $K = k\Lambda$ is a semigroup algebra contained in $k[t]$ with quotient field $k(t)$. Since $k[t]/K$ is a finitely generated K-module which is torsion with respect to $C = \{t^\lambda \mid \lambda \in \Lambda\}$, it follows that $t^n k[t] \subseteq K$ for some $n \geqslant 0$. Hence the set $\Lambda' = \mathbf{N} \setminus \Lambda$ is finite. We set $K' = \oplus_{\lambda \in \Lambda'} kt^\lambda$ so K' is a subspace of $k[t]$ with $k[t] = K \oplus K'$. For all $\lambda \in \mathbf{N}$, we have $qp \cdot t^\lambda = \lambda t^\lambda$. If we set $f(qp) = \prod_{\lambda \in \Lambda'}(qp - \lambda)$, then $f(qp) \cdot K' = 0$, so $f(qp) \in I$. Also for $\lambda \in \Lambda$, $f(qp) \cdot t^\lambda$ is a nonzero scalar multiple of t^λ, so $f(qp) \cdot K = K$. It follows that $Ik[t] = K$.

We claim that $S \subseteq D(K)$. If $t^n k[t] \subseteq K$ as above, then $q^n \in I$, so if $x \in S$, then $xq^n \in I \subseteq A_1$. Since $C' = \{q^m \mid m \geqslant 0\}$ is an Ore set in A_1, $x \in (A_1)_{C'} = k[q, q^{-1}, p] = D(k[t, t^{-1}])$. Therefore, S acts on $k[t, t^{-1}]$. Also, if $x \in S$, then $x \cdot K = x \cdot (Ik[t]) = (xI) \cdot k[t] \subseteq I \cdot k[t] \subseteq K$. Hence $x \in D(K)$. We have shown that $D(K) = S$, and S is isomorphic to the endomorphism ring of the right ideal I of A_1. Since A_1 is hereditary by [3], and simple, I is a progenerator. Therefore, $D(K)$ is Morita equivalent to A_1.

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