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LIMIT BOUNDARY VALUE PROBLEMS OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We establish necessary and sufficient conditions assuring the existence and uniqueness of solutions of the limit boundary value problems on a half-line $[a, \infty)$ for the retarded functional equation

$$\ddot{x}(t) + f(t, x(g_1(t)), \ldots, x(g_m(t)))h(\dot{x}(t)) = 0.$$

1. Introduction. We consider the retarded functional differential equation

$$\ddot{x}(t) + f(t, x(g_1(t)), \ldots, x(g_m(t)))h(\dot{x}(t)) = 0, \tag{E}$$

where

(i) f: $[t_0, \infty) \times (0, \infty)^m \to [0, \infty)$, $t_0 \in R = (-\infty, \infty)$, is a continuous function satisfying $f(t, u_1, \ldots, u_m) \geqslant f(t, v_1, \ldots, v_m)$ for any $t \geqslant t_0$ and u_j, $v_j \in (0, \infty)$, $u_j \leqslant v_j$, $j = 1, \ldots, m$;

(ii) h: $R \to (0, \infty)$ is a continuous function;

(iii) g_j: $[t_0, \infty) \to R$ are continuous functions with $g_j(t) \leqslant t$ and $g_j(t) \to \infty$ as $t \to \infty$, $j = 1, \ldots, m$;

(iv) $\int_0^\infty ds/h(s) = \infty$.

For any $\sigma \geqslant t_0$, define

$$r(\sigma) = \sigma - \min_{s \geqslant \sigma, 1 \leqslant j \leqslant m} g_j(s) \quad \text{and} \quad C_\sigma = C([-r(\sigma), 0], (0, \infty))$$

($C_\sigma = (0, \infty)$ for $r(\sigma) = 0$). We suppose for any $\phi \in C_\sigma$ and $\beta \in R$ the solution of the initial value problem (E),

$$x(t) = \phi(t - \sigma), \qquad \sigma - r(\sigma) \leqslant t \leqslant \sigma \tag{1}$$

and

$$\dot{x}(\sigma) = \beta, \tag{2}$$

uniquely exists on some interval $[\sigma - r(\sigma), \omega)$, where $\sigma < \omega \leqslant \infty$ and $[\sigma - r(\sigma), \omega)$ is the maximal interval of the existence for this solution. Thus, it is well known that the solution is continuously dependent on the initial data. If $\omega = \infty$, then the solution is called a proper solution of (E). If $\omega < \infty$, then the solution is called a nonproper solution.

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Let $x(t)$ be a proper solution on $[\sigma - r(\sigma), \infty)$, $\sigma \geqslant t_0$, and $x(\infty) = \lim_{t\to\infty} x(t)$, $\dot{x}(\infty) = \lim_{t\to\infty} \dot{x}(t)$. Since $0 \leqslant \dot{x}(\infty) < \infty$ and $0 < x(\infty) \leqslant \infty$, we see that either

(A) $\dot{x}(\infty) = \lambda > 0,\ x(\infty) = \infty$;

(B) $\dot{x}(\infty) = 0,\ x(\infty) = \text{const} > 0$; or

(C) $\dot{x}(\infty) = 0,\ x(\infty) = \infty$.

For any $\sigma \geqslant t_0$, $\phi \in C_\sigma$, and $\lambda > 0$, we shall give the necessary and sufficient condition of the existence and uniqueness for a solution of each of the following limit boundary value problems (LBVPs): (E)(1)(A), (E)(1)(B), and (E)(1)(C). All these conditions are of integral type and easy to verify. The first integral condition which is necessary and sufficient for a nonlinear second-order ordinary differential equation to have a nonoscillatory solution was obtained by F. V. Atkinson in [**1**]. There are a number of papers concerned with the boundary value problems on a half-line $[t_0, \infty)$ for ordinary differential equations [**2–5**]. Taliaferro [**6**] studied positive proper solutions of the differential equation

$$y'' + \phi(t)y^{-\lambda} = 0, \qquad \lambda > 0,$$

derived from the boundary layer equations. However, the limit boundary value problems of functional differential equations have rarely been studied.

We need the following lemma.

LEMMA. *For any $\sigma \geqslant t_0$, $\phi \in C_\sigma$, $\beta_1 < \beta_2$, let $x_1(t)$ and $x_2(t)$ be the solutions of the problems* (E)(1)(2) *with $\beta = \beta_1$ and β_2 in* (2) *respectively. Suppose both $x_1(t)$ and $x_2(t)$ exist on the same interval $[\sigma - r(\sigma), \omega)$, $\omega > \sigma$. Then we have*

$$x_1(t) < x_2(t), \qquad \sigma < t < \omega, \tag{3}$$

$$\dot{x}_1(t) < \dot{x}_2(t), \qquad \sigma \leqslant t < \omega, \tag{4}$$

and, in case $\omega = \infty$,

$$\dot{x}_1(\infty) < \dot{x}_2(\infty). \tag{5}$$

PROOF. For $\sigma \leqslant t < \omega$, from (E) we have

$$\int_{\dot{x}_1(t)}^{\dot{x}_2(t)} \frac{ds}{h(s)} = \int_{\beta_1}^{\beta_2} \frac{ds}{h(s)} + \int_\sigma^t [f(s, x_1(g_1(s)), \ldots, x_1(g_m(s))) - f(s, x_2(g_1(s)), \ldots, x_2(g_m(s)))]\, ds.$$

Inequalities (3)–(5) are now immediate.

2. Main results.

THEOREM 1. *For any $\sigma \geqslant t_0$, $\phi \in C_\sigma$, and $\lambda > 0$, there is a unique solution of the LBVP* (E)(1)(A) *if and only if for any $\varepsilon > 0$,*

$$\int^\infty f(t, (\lambda + \varepsilon)g_1(t), \ldots, (\lambda + \varepsilon)g_m(t))\, dt < \infty. \tag{6}$$

PROOF. *Necessity*. Let $x(t)$ be the solution of the LBVP (E)(1)(A). Since $\dot{x}(\infty) = \lambda$, there is a $t_1 \geqslant \sigma$ such that

$$\int_t^\infty f(s, x(g_1(s)), \ldots, x(g_m(s)))h(\dot{x}(s))\, ds < \frac{1}{2}\varepsilon$$

for $t \geqslant t_1$. If

$$t_2 = \max\{t_1, \tfrac{2}{\varepsilon}[x(t_1) - (\lambda + \tfrac{1}{2}\varepsilon)t_1]\},$$

then $x(t) \leqslant (\lambda + \varepsilon)t$ for $t \geqslant t_2$. Let $\mu = \min h(y)$, $\lambda \leqslant y \leqslant \lambda + \frac{1}{2}\varepsilon$, and $t_3 \geqslant t_2$ be such that $g_j(t) > |t_2|$ for $t \geqslant t_3$, $j = 1, \ldots, m$. The monotonicity of the function f leads us to the desired inequalities

$$\int_{t_3}^{\infty} f(t, (\lambda + \varepsilon)g_1(t), \ldots, (\lambda + \varepsilon)g_m(t))\, dt$$
$$\leqslant \frac{1}{\mu}\int_{t_3}^{\infty} f(t, x(g_1(t)), \ldots, x(g_m(t)))h(\dot{x}(t))\, dt < \infty.$$

Sufficiency. Let $x(t, \beta)$ denote the solution of the problem (E)(1)(2), $L = \{\beta \in R$: $\exists t \geqslant \sigma$ such that $\dot{x}(t, \beta) < 0$ or $\dot{x}(\infty, \beta) < \lambda\}$, and $U = \{\beta \in R$: $\dot{x}(\infty, \beta) > \lambda\}$. We shall prove both L and U are nonempty open sets.

Obviously, $L \neq \varnothing$. By the Lemma $\beta \in L$ implies $(-\infty, \beta] \subset L$. For any $\beta \in L$, if there is a $t' \geqslant \sigma$ such that $\dot{x}(t', \beta) < 0$, then, by the continuous dependence of solutions on the initial data, there is a $\beta_1 > \beta$ such that $\dot{x}(t', \beta_1) < 0$. Therefore, $\beta_1 \in L$. If β is such that the limit $\dot{x}(\infty, \beta) = \lambda_1$ exists and $\lambda_1 < \lambda = \lambda_1 + 3\delta$, then, since there is a $t_4 \geqslant \sigma$ such that $\dot{x}(t_4, \beta) \leqslant \lambda_1 + \delta$, there is a $\beta_1 > \beta$ such that $\dot{x}(t_4, \beta_1) < \lambda_1 + 2\delta$. The Lemma implies that $x(t, \beta_1)$ is a proper solution and $\dot{x}(\infty, \beta_1) \leqslant \lambda_1 + 2\delta < \lambda$. Again, we obtain $\beta_1 \in L$ and β is an interior point of L. So, L is an open set.

To prove $U \neq \varnothing$, we define a function $\psi(t)$ by

$$\begin{aligned} \psi(t) &= \phi(t - \sigma), \qquad \sigma - r(\sigma) \leqslant t \leqslant \sigma, \\ &= \phi(0) + (\lambda + 1)(t - \sigma), \qquad t > \sigma. \end{aligned} \tag{7}$$

In view of conditions (iv) and (6), we can choose $\beta > 0$ so large that

$$H(\beta) > H(\lambda + 1) + \int_{\sigma}^{\infty} f(s, \psi(g_1(s)), \ldots, \psi(g_m(s)))\, ds,$$

where $H(y) = \int_0^y ds/h(s)$. Let $x(t)$ be the solution of (E)(1)(2). We claim $\dot{x}(t) > \lambda + 1$ for $t \geqslant \sigma$. If this is not true, then there is a $t_5 > \sigma$ such that $\dot{x}(t) > \lambda + 1$, $\sigma \leqslant t \leqslant t_5$, and $\dot{x}(t_5) = \lambda + 1$. Here we have $x(t) \geqslant \psi(t)$ for $\sigma \leqslant t \leqslant t_5$ and $x(t) = \psi(t)$ for $\sigma - r(\sigma) \leqslant t \leqslant \sigma$, and

$$H(\dot{x}(t_5)) = H(\beta) - \int_{\sigma}^{t_5} f(s, x(g_1(s)), \ldots, x(g_m(s)))\, ds$$
$$\geqslant H(\beta) - \int_{\sigma}^{t_5} f(s, \psi(g_1(s)), \ldots, \psi(g_m(s)))\, ds > H(\lambda + 1).$$

This contradicts the definition of t_5 and proves $\beta \in U \neq \varnothing$.

For any $\beta_3 \in U$, by the Lemma we have $[\beta_3, \infty) \subset U$. Suppose $\dot{x}(\infty, \beta_3) = \lambda_3 > \lambda = \lambda_3 + 3\eta$, $\eta > 0$. Choose $t_6 > \sigma$ large enough that

$$g_j(t) > |(\lambda + 2\eta)\sigma - \phi(0)|/\eta, \qquad j = 1, \ldots, m, \tag{8}$$

for $t \geqslant t_6$ and

$$\int_{t_0}^{\infty} f(s, (\lambda + \eta)g_1(s), \ldots, (\lambda + \eta)g_m(s))\, ds < H(\lambda + 3\eta) - H(\lambda + 2\eta). \tag{9}$$

Since $\dot{x}(t_6, \beta_3) > \lambda_3$, we can find a $\beta_4 < \beta_3$ such that $\dot{x}(t_6, \beta_4) \geqslant \lambda + 3\eta$. We claim that $\dot{x}(t, \beta_4) > \lambda + 2\eta$ for $t \geqslant \sigma$; hence, $\beta_4 \in U$. Suppose not. Then there is a $t_7 > t_6$ such that $\dot{x}(t, \beta_4) > \lambda + 2\eta$ for $\sigma \leqslant t < t_7$ and $\dot{x}(t_7, \beta_4) = \lambda + 2\eta$. From (8) we get

$$x\big(g_j(t), \beta_4\big) \geqslant \phi(0) + (\lambda + 2\eta)\big(g_j(t) - \sigma\big) \geqslant (\lambda + \eta) g_j(t), \qquad j = 1, \dots, m, \tag{10}$$

whenever $t_6 \leqslant t \leqslant t_7$. According to (9), (10), and (i) we obtain

$$\begin{aligned} H\big(\dot{x}(t_7, \beta_4)\big) &= H\big(\dot{x}(t_6, \beta_4)\big) - \int_{t_6}^{t_7} f\big(s, x(g_1(s), \beta_4), \dots, x(g_m(s), \beta_4)\big)\, ds \\ &\geqslant H(\lambda + 3\eta) - \int_{t_6}^{t_7} f\big(s, (\lambda + \eta) g_1(s), \dots, (\lambda + \eta) g_m(s)\big)\, ds \\ &> H(\lambda + 2\eta), \end{aligned}$$

which contradicts the definition of t_7. This shows that $\beta_4 \in U$ and U is an open set.

Finally, the set $B = \{\beta \in R\colon\ \dot{x}(\infty, \beta) = \lambda\}$ is nonempty and, by the Lemma, contains only a single point. This completes the proof of Theorem 1.

COROLLARY. *For any $\sigma \geqslant t_0$, $\phi \in C_\sigma$, there exists a unique solution of the problem* (E)(1) *and $\dot{x}(\infty) = 0$ if and only if for any $\varepsilon > 0$,*

$$\int^{\infty} f\big(t, \varepsilon g_1(t), \dots, \varepsilon g_m(t)\big)\, dt < \infty. \tag{11}$$

In fact, the proof may be carried out similarly to that of Theorem 1 except we set $\lambda = 0$ in (6) and $L = \{\beta \in R\colon\ \exists t \geqslant \sigma$ such that $\dot{x}(t, \beta) < 0\}$ and $U = \{\beta \in R\colon\ \dot{x}(\infty, \beta) > 0\}$ in the sufficiency part of the proof.

EXAMPLE 1. Consider the following equation:

$$\ddot{x}(t) = -\exp[t - x(t-1)]/(t-1), \qquad t \geqslant 2. \tag{12}$$

Since (6) is valid for $\lambda = 1$ and any $\varepsilon > 0$, by Theorem 1, for any $\sigma \geqslant 2$, $\phi \in C([-1, 0], (0, \infty))$, there exists a unique solution of (12) satisfying $x(t) = \phi(t - \sigma)$, $\sigma - 1 \leqslant t \leqslant \sigma$, $\dot{x}(\infty) = 1$. Note that (6) fails to be valid for $\lambda = 1$ and $\varepsilon = 0$.

EXAMPLE 2. For

$$\ddot{x}(t) = -\big[1 + \dot{x}^2(t)\big] x^{-2}(t - \pi/2) \tag{13}$$

condition (6) is true, but Theorem 1 fails to hold because condition (iv) is false. In fact, for any $\beta \in R$ the solution $x(t)$ of (13) satisfying $x(t) = 1$, $-\pi/2 \leqslant t \leqslant 0$, and $\dot{x}(0) = \beta$ is nonproper, since from (13) we have $\arctan \dot{x}(t) = \arctan \beta - t$ for $0 < t \leqslant \pi/2$ and $\dot{x}(t)$ vanishes at $t = \arctan \beta$.

THEOREM 2. *For any $\sigma \geqslant t_0$, $\phi \in C_\sigma$, there exists a unique solution of the LBVP* (E)(1)(B) *if and only if for some positive number a,*

$$\int_{t_0}^{\infty} t f(t, a, \dots, a)\, dt < \infty. \tag{14}$$

PROOF. Let $x(t)$ be the solution of the LBVP (E)(1)(B), $x(\infty) = a > 0$. Let $\mu = \min h(y)$, $0 \leqslant y \leqslant \dot{x}(\sigma) = \beta$. We select an $s \geqslant \sigma$ large enough that $g_j(t) \geqslant \sigma$ for $t \geqslant s$, $j = 1, \dots, m$. Then, by condition (i) and the fact that $\phi(0) \leqslant x(t) \leqslant a$ and $0 \leqslant \dot{x}(t) \leqslant \beta$ for all $t \geqslant \sigma$, (14) may be obtained from the following estimates:

$$\begin{aligned}\mu \int_s^\infty (t-s) f(t, a, \dots, a)\, dt & \\ &\leqslant \int_s^\infty (t-s) f(t, x(g_1(t)), \dots, x(g_m(t))) h(\dot{x}(t))\, dt \\ &= a - x(t) < \infty.\end{aligned}$$

From conditions (i) and (iii), we observe that (14) implies (11). By the Corollary of Theorem 1, there is a unique solution $x(t)$ of (E) satisfying (1) and $\dot{x}(\infty) = 0$. It remains to show that $x(\infty)$ is a finite number. To do this we need only verify that $x(t)$ is bounded since $x(t)$ is increasing for $t \geqslant \sigma$. If $x(t)$ is unbounded, then $x(\infty) = \infty$ and $x(g_j(t)) \geqslant a$, $j = 1, \dots, m$, for t large, say, $t \geqslant t_1 \geqslant \sigma$. Since $\dot{x}(\infty) = 0$, we have

$$\dot{x}(t) = \int_t^\infty f(s, x(g_1(s)), \dots, x(g_m(s))) h(\dot{x}(s))\, ds \leqslant M \int_t^\infty f(s, a, \dots, a)\, ds$$

for $t \geqslant t_1$, where $M = \max h(y)$, $0 \leqslant y \leqslant \dot{x}(\sigma)$. So, for any t' and t'', $t'' > t' \geqslant t_1$, we get

$$x(t'') - x(t') = \int_{t'}^{t''} \dot{x}(s)\, ds \leqslant M \int_{t'}^\infty s f(s, a, \dots, a)\, ds.$$

It follows from (14) that $x(\infty)$ is a finite number. Thus we are led to a desired contradiction, which proves the theorem.

THEOREM 3. *For any $\sigma \geqslant t_0$, $\phi \in C_\sigma$, there exists a unique solution of the LBVP* (E)(1)(C) *if and only if for any $\varepsilon > 0$ and $a > 0$ condition* (11) *and*

$$\int_{t_0}^\infty t f(t, a, \dots, a)\, dt = \infty \tag{15}$$

are satisfied.

Theorem 3 follows directly from the Corollary of Theorem 1 and Theorem 2.

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