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REMARK ON "TRIDIAGONAL MATRIX REPRESENTATIONS OF CYCLIC SELF-ADJOINT OPERATORS"

PENG FAN

ABSTRACT. In [2], Dombrowski used a "orthogonal polynomials" technique to obtain a sufficient condition (in terms of the weights) for the existence of an absolutely continuous subspace for real parts of unilateral weighted shifts. The purpose of this note is to present a "diagonal" technique that produces a trace criterion for the existence of an absolutely continuous subspace for real parts as well as unitary parts of (bounded) operators.

Let $B(H)$ denote the class of all (bounded) linear operators acting on a complex separable Hilbert space H. For $T \in B(H)$, we define $\operatorname{Re} T = (T + T^*)/2$, $[T^*, T] = T^*T - TT^*$ and $|T| = (T^*T)^{1/2}$. For the hermitian operator A, we let $A^{\pm} = (|A| \pm A)/2$ and $H_a(A) = \{x \mid \|E(\lambda)x\|^2$ is absolutely continuous as a function of λ, where E is the spectral measure of $A\}$. Observe that $H_a(A)$ is a reducing subspace for A (see, e.g., [5, p. 104]). $H_a(A)$ is called the absolutely continuous subspace of A; its orthogonal complement, denoted by $H_s(A)$, is called the singular subspace of A. Similarly, we can define $H_a(U)$ and $H_s(U)$ as the absolutely continuous subspace and singular subspace, respectively, of a unitary operator U.

In [2], Dombrowski used a "orthogonal polynomials" technique to obtain a sufficient condition (in terms of the weights) for the existence of an absolutely continuous subspace for real parts of unilateral weighted shifts. The purpose of this note is to present a "diagonal" technique that produces a trace criterion for the existence of an absolutely continuous subspace for real parts as well as unitary parts of (bounded) operators. Different as it may look, this criterion turns out to be equivalent to that of Dombrowski's when the operators are unilateral weighted shifts; besides, in the case of bilateral weighted shifts, it can easily be translated into a Dombrowski type condition. Furthermore, the technique leads to a refinement of a result of Putnam [6, Theorem 2.4.1 (ii)].

1. The main results. Before proceeding to the main results, we list the following tools for convenience.

(I) Every singular hermitian operator can be written as a sum of a diagonal operator and a trace class operator; see Carey and Pincus [1, p. 484].

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(II) Every singular unitary operator can be written as a sum of a diagonal unitary operator and a trace class operator; see [**1**, p. 509].

(III) Let A be a hermitian operator. Then A has a zero diagonal (i.e., there exist an orthonormal basis $\{b_n\}$ such that $(Ab_n, b_n) = 0$ for all n) if and only if $\operatorname{tr} A^+ = \operatorname{tr} A^- \leqslant \infty$; see Fan [**3**].

(IV) Let $T \in B(H)$. If there exists an orthonormal basis $\{b_n\}$ such that $\Sigma(Tb_n, b_n) = 0$, then T has a zero diagonal; see [**3**].

THEOREM 1. *Let $T \in B(H)$. If* $\operatorname{tr}[T^*, T]^+ \neq \operatorname{tr}[T^*, T]^-$, *then* $\operatorname{Re} T$ *has an absolutely continuous part.*

PROOF. Write $T = A + iB$ where A and B are hermitian (of course $A = \operatorname{Re} T$); then $[T^*, T] = 2i(AB - BA)$. Suppose A is singular. We shall prove that $\operatorname{tr}[T^*, T]^+ = \operatorname{tr}[T^*, T]^-$. Indeed, by (I), $A = D + C$ where C is of trace class and $D = \operatorname{diag}\{d_1, d_2, \dots\}$ with respect to some orthonormal basis $\{b_n\}$; hence $[T^*, T] = 2i[(DB - BD) + (CB - BC)]$. It is a folk theorem that $\operatorname{tr}(CB - BC) = 0$. As for $DB - BD$, this term has a zero diagonal. In fact, $((DB - BD)b_n, b_n) = d_n(Bb_n, b_n) - d_n(Bb_n, b_n) = 0$ for all n. Therefore, the sum of the entries of $[T^*, T]$ with respect to $\{b_n\}$ is zero. By (IV), $[T^*, T]$ has a zero diagonal; hence by (III), $\operatorname{tr}[T^*, T]^+ = \operatorname{tr}[T^*, T]^-$. This completes the proof.

A result of Putnam [**6**, Theorem 2.4.1(ii)] says that if $C = AB - BA$ where A is singular and hermitian or unitary then 0 belongs to the "interior" of the numerical range $W(C)$ of C. (Here a complex number z is said to belong to the "interior" of a convex set W if z is in W and if one of the following conditions holds: if W is two dimensional, z is an interior point; if W is a line segment, z is not an end-point; finally, W consists of the single point z.) This result can be refined as follows:

COROLLARY. *If $C = AB - BA$ where A is singular and hermitian or unitary, then C has a zero diagonal.*

Note that if C has a zero diagonal then 0 belongs to the "interior" of $W(C)$, but not conversely; see [**3**, Corollary 3].

THEOREM 2. *Let $T \in B(H)$. Suppose $T = UP$ where U is unitary and P is positive. If* $\operatorname{tr}[T^*, T]^+ \neq \operatorname{tr}[T^*, T]^-$, *then U has an absolutely continuous part.*

PROOF. Suppose U is singular. By (II), $U = D + C$ where C is of trace class and D is diagonal and unitary. Now

$$\begin{aligned}[T^*, T] &= PU^*UP - UP^2U^* \\ &= (P^2 - DP^2D^*) + (PD^*CP - CPPD^*) + (PC^*DP - DPPC^*) \\ &\quad + (PC^*CP - CPPC^*).\end{aligned}$$

It is easy to see that the first term has a zero diagonal and that the traces of the last three terms are all zero (because CP is of trace class). Thus, by (IV) and (III), $\operatorname{tr}[T^*, T]^+ = \operatorname{tr}[T^*, T]^-$. The proof is complete.

REMARKS. (1) It is easy to see that the converses of Theorems 1 and 2 are not valid.

(2) Immediate consequences of the criterion are the facts that real parts and unitary parts of pure hyponormal operators have an absolutely continuous subspace. It is perhaps pertinent to mention here that some structure theorems of hyponormal operators can be proved directly, basing on the arguments in the proofs of Theorems 1 and 2; see **[4]**.

(3) Also pointed out here should be the fact that there are more self-commutators S with $\operatorname{tr} S^+ = \operatorname{tr} S^-$ than those with $\operatorname{tr} S^+ \neq \operatorname{tr} S^-$, in the sense that self-commutators with $\operatorname{tr} S^+ = \operatorname{tr} S^-$ are norm dense in the class of all self-commutators (see **[3]**), whereas those with $\operatorname{tr} S^+ \neq \operatorname{tr} S^-$ are only strong dense in that class (because $I \oplus (-I)$ does not belong to the norm closure of self-commutators with $\operatorname{tr} S^+ \neq \operatorname{tr} S^-$).

2. Applications to (unilateral and bilateral) shifts.

THEOREM 3. *Let T be a unilateral shift with the positive weight sequence $\{a_n\}_{n=1}^{\infty}$. Then the following statements are equivalent*:

(1) $\operatorname{tr}[T^*, T]^+ \neq \operatorname{tr}[T^*, T]^-$,

(2) $\{a_n\}$ *converges to a strictly positive limit and* $[T^*, T]$ *is of trace class*,

(3) $\{a_n\}$ *converges to a strictly positive limit and* $\sum |a_{n+1} - a_n| < \infty$.

PROOF. The key to the equivalence lies in the simple observation that $[T^*, T] = \operatorname{diag}\{a_1^2, a_2^2 - a_1^2, \dots\}$.

(2) $\Rightarrow$ (1). If $\lim a_n = a > 0$ and $[T^*, T]$ is of trace class, then $\operatorname{tr}[T^*, T] = \lim a_n^2 = a^2 > 0$. Hence $\operatorname{tr}[T^*, T]^+ \neq \operatorname{tr}[T^*, T]^-$.

(1) $\Rightarrow$ (2). If not (2), then three possibilities need to be checked.

(i) $\lim a_n$ does not exist: In this case $\operatorname{tr}[T^*, T]^+ = \operatorname{tr}[T^*, T]^- = \infty$.

(ii) $\lim a_n = 0$: The hypothesis of (IV) is satisfied. By (IV) and (III), $\operatorname{tr}[T^*, T]^+ = \operatorname{tr}[T^*, T]^- \leq \infty$.

(iii) $\lim a_n = a > 0$ but $[T^*, T]$ is not of trace class: It is obvious that $\operatorname{tr}[T^*, T]^+ = \operatorname{tr}[T^*, T]^- = \infty$.

(2) $\Leftrightarrow$ (3). An elementary series convergence argument.

The proof is now complete.

Thus the following result of Dombrowski **[2]** is immediate.

COROLLARY. *Let T be as above. If $\{a_n\}$ converges to a strictly positive limit and $[T^*, T]$ is of trace class, then* $\operatorname{Re} T$ *has an absolutely continuous part.*

It should be noted here that the "orthogonal polynomials" technique provides a further clue which our technique lacks; namely, it also shows that the spectrum of the absolutely continuous part contains $(-a, a)$.

Finally, we consider the bilateral shifts case. Since this case differs only slightly from the unilateral shifts case, the proof will be omitted.

THEOREM 4. *Let T be a bilateral shift with the positive weight sequence $\{a_n\}_{n=-\infty}^{\infty}$. Then* $\operatorname{tr}[T^*, T]^+ \neq \operatorname{tr}[T^*, T]^-$ *if and only if $\{a_n\}$ and $\{a_{-n}\}$ converge to two distinct numbers and $[T^*, T]$ is of trace class. In case $\{a_n\}$ and $\{a_{-n}\}$ converge to two distinct numbers other than* 0, $[T^*, T]$ *is of trace class if and only if* $\sum |a_{n+1} - a_n| < \infty$.

ADDED IN PROOF. In [*Tridiagonal matrix representations of cyclic self-adjoint operators*. II, Pacific J. Math. **120** (1985), 47–53], Dombrowski extended her result to hermitian operators that are real parts of operators T defined by $Te_n = b_n e_n + 2a_n e_{n+1}$ where $\{e_n\}$ is an orthonormal basis. An easy computation shows that $\operatorname{tr}[T^*, t] = 4 \lim a_n^2 = 4a^2$; hence, by Theorem 1 above, $\operatorname{Re} T$ has an absolutely continuous part (here a is assumed to be positive).

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DEPARTMENT OF MATHEMATICS, TEXAS CHRISTIAN UNIVERSITY, FORT WORTH, TEXAS 76129