PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 2, October 1986

THE K-FUNCTIONAL FOR H^p AND BMO IN THE POLY-DISK[1]

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ABSTRACT. Peetre's K-functional for the Hardy space H^p, $0 < p < +\infty$, and the space BMO of functions of bounded mean oscillation is explicitly characterized in the case of a product of upper half-spaces.

1. Introduction. In this note we consider the K-functional for the Hardy spaces H^p, $0 < p < +\infty$, and the space BMO of functions of bounded mean oscillation in the case of a product of upper half-spaces. The main result in **[J]** is a characterization of the K-functional for H^p and BMO in $\mathbf{R}^n$ in terms of a certain truncated square function, and here we prove the analogous result in the product case. Our proof is based on a refinement of some ideas in Chang-Fefferman **[C-F]** and uses a Calderón-Zygmund type procedure where certain families of open sets play the role of the dyadic cubes in the classical case.

Let us briefly recall some relevant facts and definitions (see **[C-F** and **B-L]** for more details):

In what follows we shall, for simplicity, work with the domain $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ and its distinguished boundary $\mathbf{R}^2$. Points in $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ are denoted by (y, t) where $y \in \mathbf{R}^2$ and $t = (t_1, t_2)$, $t_1, t_2 > 0$. The notation $\varphi(u)$ is reserved for an even, real-valued, $C^\infty(\mathbf{R})$ function with support in $[-1, 1]$ and such that

$$\int_0^\infty \hat{\varphi}(u)^2 \frac{du}{u} = 1, \qquad \left(\frac{d}{du}\right)^m \hat{\varphi}(0) = 0$$

for sufficiently large m to be specified. Given such a φ, we let $\Phi_t(y) = \varphi(y_1/t_1)\varphi(y_2/t_2)/t_1t_2$. If f is a tempered distribution, we put $f(y,t) = f * \Phi_t(y)$, and define the double square function Sf by

$$Sf(x) = \left(\iint_{\Gamma(x)} |f(y,t)|^2 dy \, \frac{dt_1 dt_2}{t_1^2 t_2^2} \right)^{1/2},$$

$x = (x_1, x_2) \in \mathbf{R}^2$. Here $\Gamma(x)$ denotes the product cone

$$\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2) = \left\{ (y,t) \colon |x_1 - y_1| < t_1, |x_2 - y_2| < t_2 \right\}.$$

Received by the editors September 9, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42B30.
Key words and phrases. H^p-space, atomic decomposition, K-functional, poly-disk.
[1]Research supported in part by a grant from NSF.

If $m \sim (1/p - 1)_+$, the Hardy space $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, $0 < p < +\infty$, can be defined (modulo a normalization) as the set of tempered distributions f such that

$$\| f \|_{H^p} = \| Sf \|_{L^p(\mathbf{R}^2)} < +\infty .$$

Now let $R_{y,t}$ denote the rectangle centered at y with dimensions $t_1 \times t_2$. With each open set $\Omega \subset \mathbf{R}^2$ we associate the Carleson region $C(\Omega) = \{(y,t)\colon R_{y,t} \subset \Omega\}$. The space $\mathrm{BMO}(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ of functions of bounded mean oscillation is the set of all tempered distributions f (modulo a suitable class $\mathscr{A}$ of distributions with Fourier-transforms supported on the coordinates axes) such that

$$\| f \|_{\mathrm{BMO}} = \sup_{\Omega} \left(\frac{1}{|\Omega|} \iint_{C(\Omega)} |f(y,t)|^2 \, dy \, \frac{dt_1 dt_2}{t_1 t_2} \right)^{1/2} < +\infty .$$

According to Chang-Fefferman **[C-F]**, $\mathrm{BMO}(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ can be identified with the dual of $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$. Moreover,

$$\| f \|_{(H^1)^*} \approx \| f \|_{\mathrm{BMO}}. \tag{1.1}$$

Finally, we write down the definition of Peetre's K-functional for the couple (H^p, BMO), $0 < p < +\infty$:

$$K(t,f; H^p, \mathrm{BMO}) = \inf_{f = f_0 + f_1} \max\bigl(\| f_0 \|_{H^p}, t \| f_1 \|_{\mathrm{BMO}} \bigr)$$

defined for $f \in H^p + \mathrm{BMO}$ and $t > 0$.

2. Some auxiliary facts. A countable family $\mathscr{O} = \{\Omega_i\}$ of open sets $\Omega_i \subset \mathbf{R}^2$ of finite measure is called *directed* if either $|\Omega_i \cap \Omega_j| = 0$, $\Omega_i \subset \Omega_j$ or $\Omega_j \subset \Omega_i$ whenever $\Omega_i, \Omega_j \in \mathscr{O}$. With such a family $\mathscr{O}$ we associate the maximal operator $M_{\mathscr{O}}$ defined by

$$M_{\mathscr{O}} f(x) = \sup_{x \in \Omega_i} \frac{1}{|\Omega_i|} \int_{\Omega_i} |f(y)| \, dy, \qquad \Omega_i \in \mathscr{O},$$

whenever $x \in \bigcup_i \Omega_i$, $M_{\mathscr{O}} f(x) = 0$ otherwise. In a standard fashion, it follows that $M_{\mathscr{O}}$ is of weak-type $(1,1)$ and bounded on L^p, $1 < p \leqslant +\infty$:

$$t |\{ M_{\mathscr{O}} f > t \}| \leqslant \| f \|_{L^1}, \tag{2.1}$$

$$\| M_{\mathscr{O}} f \|_{L^p} \leqslant c_p \| f \|_{L^p}, \qquad 1 < p \leqslant +\infty . \tag{2.2}$$

We also introduce the corresponding "local maximal operators" $M_{\alpha,\mathscr{O}}$, $0 < \alpha < 1$, defined by

$$M_{\alpha,\mathscr{O}} f(x) = \sup_{x \in \Omega_i} \inf\bigl\{ A\colon \bigl|\{ y \in \Omega_i\colon |f(y)| > A \}\bigr| < \alpha |\Omega_i| \bigr\}, \qquad \Omega_i \in \mathscr{O},$$

cf. **[J-T]**. (In particular, when $\alpha = \frac{1}{2}$. $M_{\alpha,\mathscr{O}} f(x)$ is the supremum of the median-values of f over the sets Ω_i containing x.) Observing that $\{ M_{\alpha,\mathscr{O}} f > t \} = \{ M_{\mathscr{O}} \chi_{|f| > t} \geqslant \alpha \}$, (2.1) gives us

LEMMA 2.1. *Let $0 < \alpha < 1$. If $\mathscr{O}$ is a directed family of open sets, then*

$$|\{ M_{\alpha,\mathscr{O}} f > t \}| \leqslant \bigl|\{ |f| > t \}\bigr| / \alpha, \qquad t > 0.$$

Let $\Omega \subset \mathbf{R}^2$ be an open set and let $\Gamma_\Omega(x)$, $x \in \mathbf{R}^2$, denote the truncated cone

$$\Gamma_\Omega(x) = \left\{(y,t) \in \mathbf{R}^2_+ \times \mathbf{R}^2_+ \colon |x_i - y_i| < t_i,\ i = 1,2, \text{ and } |R_{y,t} \cap \Omega| > \tfrac{1}{100}|R_{y,t}|\right\}.$$

Following Chang-Fefferman [C-F], we put

$$S_\Omega f(x) = \left(\int\!\!\int\!\!\int_{\Gamma_\Omega(x)} |f(y,t)|^2\, dy \frac{dt_1 dt_2}{t_1^2 t_2^2}\right)^{1/2}.$$

(Recall that $f(y,t)$ is a convolution, $f * \Phi_t(y)$; notice, however, that very little of the argument below uses this fact.) If $\mathcal{O} = \{\Omega_i\}$ is a directed family of open sets, the "local square functions" $S_{\alpha,\mathcal{O}}$, $0 < \alpha < 1$, are defined by

$$S_{\alpha,\mathcal{O}} f(x) = \sup_{x \in \Omega_i} \inf\left\{A \colon \left|\{y \in \Omega_i \colon S_{\Omega_i} f(y) > A\}\right| < \alpha|\Omega_i|\right\}, \qquad \Omega_i \in \mathcal{O}.$$

The particular value of α is not important; the main reason for introducing it is that it makes it easier to express the subadditivity property

$$S_{\alpha,\mathcal{O}}(f+g)(x) \leqslant 2\big(S_{\alpha/2,\mathcal{O}} f(x) + S_{\alpha/2,\mathcal{O}} g(x)\big). \tag{2.3}$$

Clearly, $S_{\alpha,\mathcal{O}} f \leqslant M_{\alpha,\mathcal{O}}(Sf)$. Hence, by Lemma 2.1 and the definition of H^p,

$$\|S_{\alpha,\mathcal{O}} f\|_{L^p} \leqslant c_\alpha \|f\|_{H^p}, \qquad 0 < p < +\infty. \tag{2.4}$$

On the other hand, by Chebyshev's inequality,

$$\begin{aligned}\left|\{y \in \Omega_i \colon S_{\Omega_i} f > A\}\right| &\leqslant \int_{\Omega_i} S_{\Omega_i} f(y)^2\, dy/A^2 \\ &\leqslant c\int\!\!\int_{C(\tilde{\Omega}_i)} |f(z,t)|^2\, dz \frac{dt_1 dt_2}{t_1 t_2}/A^2,\end{aligned}$$

where $\tilde{\Omega}_i = \{M\chi_{\Omega_i} > \frac{1}{100}\}$ and M is the strong maximal operator. If $f \in \mathrm{BMO}$, this is less than

$$c\|f\|^2_{\mathrm{BMO}}|\tilde{\Omega}_i|/A^2 \leqslant c\|f\|^2_{\mathrm{BMO}}|\Omega_i|/A^2$$

by the strong maximal theorem. Hence, when $A^2 > c\|f\|^2_{\mathrm{BMO}}/\alpha$ we have $|\{y \in \Omega_i \colon S_{\Omega_i} f(y) > A\}| < \alpha|\Omega_i|$, and consequently,

$$\|S_{\alpha,\mathcal{O}} f\|_{L^\infty} \leqslant c_\alpha \|f\|_{\mathrm{BMO}}. \tag{2.5}$$

Now select, once and for all, a family $\{O_k\}_{k \in Z}$ of open sets $O_k \subset \mathbf{R}^2$ of finite measure with the following properties:

$$\bigcup_k O_k = \mathbf{R}^2, \quad \bigcap_k O_k = \{0\}, \quad \left\{M\chi_{O_{k+1}} > \tfrac{1}{4}\right\} \subset O_k, \qquad k \in Z. \tag{2.6}$$

Let $\Omega_k = O_k \setminus \overline{O}_{k+1}$ and let

$$\mathscr{R}_k = \left\{\text{all dyadic rectangles } R \text{ such that } |R \cap O_k| > \tfrac{1}{2}|R|,\ |R \cap O_{k+1}| \leqslant \tfrac{1}{2}|R|\right\}, \qquad k \in Z.$$

Observe that each dyadic rectangle belongs to exactly one $\mathscr{R}_k$.

We will need the following simple lemma.

LEMMA 2.2. *If $R \in \mathscr{R}_k$, then $|R \cap \Omega_k| \geqslant \frac{1}{4}|R|$.*

PROOF. There are two possibilities: either $|R \cap O_{k+1}| > \frac{1}{4}|R|$ or $|R \cap O_{k+1}| \leqslant \frac{1}{4}|R|$. In the first case, $R \subset \{M\chi_{O_{k+1}} > \frac{1}{4}\} \subset O_k$ by (2.6). Hence,

$$|R \cap \Omega_k| \geqslant |R \cap O_k| - |R \cap O_{k+1}| \geqslant |R| - \tfrac{1}{2}|R| \geqslant \tfrac{1}{4}|R|$$

since $R \in \mathscr{R}_k$. Similarly, in the second case,

$$|R \cap \Omega_k| \geqslant |R \cap O_k| - |R \cap O_{k+1}| \geqslant \tfrac{1}{2}|R| - \tfrac{1}{4}|R| = \tfrac{1}{4}|R|.$$

This proves the lemma.

3. The main result. We are now ready to state our main result.

THEOREM 3.1. *Let $0 < p < +\infty$ and $0 < \alpha < 1$. For each directed family $\mathcal{O}$ of open sets*

$$K(t, S_{\alpha,\mathcal{O}}f; L^p, L^\infty) \leqslant c_\alpha K(t, f; H^p, \mathrm{BMO}), \qquad t > 0.$$

Conversely, for each f there is a directed family $\mathcal{O} = \mathcal{O}_f$ such that

$$K(t, f; H^p, \mathrm{BMO}) \leqslant c_{\alpha,p} K(t, S_{\alpha,\mathcal{O}}f; L^p, L^\infty), \qquad t > 0.$$

PROOF. The first part follows immediately from (2.3)–(2.5).

Now recall that $K(t, f; L^p, L^\infty)/t$ is the inverse of the best approximation functional

$$E(t, f; L^p, L^\infty)/t = \inf_{\|f_1\|_{L^\infty} \leqslant t} \|f - f_1\|_{L^p}/t,$$

and similarly for H^p and BMO (see **[B-L, J-T]**). Furthermore, it is easy to see that $E(t, f; L^p, L^\infty) \approx (\int_{|f|>t} |f|^p)^{1/p}$. Hence, to prove the second converse part it is enough to show that there is a family $\mathcal{O} = \mathcal{O}_f$ such that

$$E(t, f; H^p, \mathrm{BMO}) \leqslant c\left(\int_{S_{\alpha,\mathcal{O}}f > t/c} |S_{\alpha,\mathcal{O}}f|^p\right)^{1/p} \tag{3.1}$$

for some constant c. By reiteration, it will be enough to do this for $0 < p < 1$.

Let us first consider the construction of the directed family $\mathcal{O}$. Let $\{O_k\}$ be the family satisfying (2.6) that we selected earlier. $\mathcal{O} = \{\Omega_{k,l}\}$, $k \in Z$, $l \in N$, will be derived from $\{O_k\}$ by an inductive argument:

We start by fixing $k \in Z$. To get the induction started we define $\Omega_{k,0} = \Omega_k = O_k \setminus \overline{O}_{k+1}$. Suppose $\Omega_{k,0}, \ldots, \Omega_{k,l-1}$ have been chosen and put

$$A_{k,l-1} = \inf\left\{A \colon \left|\left\{x \in \Omega_{k,l-1} \colon S_{\Omega_{k,l-1}}f(x) > A\right\}\right| < \alpha|\Omega_{k,l-1}|\right\}.$$

Then

$$\Omega_{kl} = \left\{x \in \Omega_{k,l-1} \colon S_{\Omega_{k,l-1}}f(x) > A_{k,l-1}\right\}.$$

We repeat this process for each k. The sets Ω_{kl} obtained in this way satisfy $\Omega_{kl} \subset \Omega_{k,l-1}$, and since the sets $\Omega_k = \Omega_{k,0}$ are pairwise disjoint, $\mathcal{O} = \{\Omega_{kl}\}$ is a directed family. Also, $S_{\alpha,\mathcal{O}}f \in L^p + L^\infty$ implies that

$$|\Omega_{kl}| \leqslant \alpha|\Omega_{k,l-1}|. \tag{3.2}$$

For $k \in Z$, $l \in N$ we now define

$$\mathscr{R}_{kl} = \left\{ R \in \mathscr{R}_k \colon |R \cap \Omega_{kl}| > \tfrac{1}{10}|R|, |R \cap \Omega_{k,l+1}| \leqslant \tfrac{1}{10}|R| \right\}.$$

By Lemma 2.2 it follows that in this way each dyadic rectangle R belongs to a unique $\mathscr{R}_{kl}$. If $R = I \times J$, we put $R^+ = \{(y,t) \in \mathbf{R}^2_+ \times \mathbf{R}^2_+ \colon y \in R, 2|I| < t_1 < 4|I|, 2|J| < t_2 < 4|J|\}$ and $B^+_{kl} = \bigcup_{R \in \mathscr{R}_{kl}} R^+$.

The almost optimal decomposition of $f = f_0 + f_1$ required to prove (3.1) is given by

$$f_0 = \sum_k \sum_{l \in L(k)} \iint_{B^+_{kl}} f(y,t) \Phi_t(x-y)\, dy \frac{dt_1 dt_2}{t_1 t_2} \equiv \sum_k \sum_{l \in L(k)} b_{kl},$$

where $L(k) = L(k,t) = \{l \colon \Omega_{kl} \subset \{S_{\alpha,\mathscr{O}} f > t\}\}$, and

$$f_1 = \sum_k \sum_{l \notin L(k)} \iint_{B^+_{kl}} f(y,t) \Phi_t(x-y)\, dy \frac{dt_1 dt_2}{t_1 t_2} \equiv \sum_k \sum_{l \notin L(k)} b_{kl}$$

(with convergence in $\mathscr{S}'/\alpha$). Notice that, for each k, $L(k)$ is either empty or there is a unique $l(k)$ such that $\Omega_{kl} \subset \{S_{\alpha,\mathscr{O}} f > t\}$ if $l \geqslant l(k)$.

By checking Fourier transforms it follows that $f = f_0 + f_1$ (in $\mathscr{S}'/\alpha$). Hence, to prove (3.1) we need to verify that

$$\|f_0\|^p_{H^p} \leqslant c \int_{S_{\alpha,\mathscr{O}} f > t} |S_{\alpha,\mathscr{O}} f|^p \tag{3.3}$$

and

$$\|f_1\|_{\mathrm{BMO}} \leqslant ct. \tag{3.4}$$

We claim that for each k and l b_{kl} is a p-atom (in the sense of **[C-F]**) after the appropriate normalization $cb_{kl}(x)/\sup_{x \in \Omega_{kl} \setminus \Omega_{k,l+1}} S_{\alpha,\mathscr{O}} f(x) |\Omega_{kl}|^{1/p}$. This would imply that

$$\begin{aligned}
\|f_0\|^p_{H^p} &\leqslant \sum_k \sum_{l \in L(k)} \|b_{kl}\|^p_{H^p} \\
&\leqslant c \sum_k \sum_{l \in L(k)} \Big(\sup_{x \in \Omega_{kl} \setminus \Omega_{k,l+1}} S_{\alpha,\mathscr{O}} f(x) \Big)^p |\Omega_{kl}|.
\end{aligned}$$

By (3.2),

$$|\Omega_{kl}| \leqslant c |\Omega_{kl} \setminus \Omega_{k,l+1}| = c \Big| \Omega_{kl} \setminus \bigcup_{m > l} \Omega_{km} \Big|$$

since $\Omega_{km} \subset \Omega_{k,l+1}$ if $m > l$. Also notice that $S_{\alpha,\mathscr{O}} f(x)$ is an elementary function which is constant on $\Omega_{kl} \setminus \Omega_{k,l+1}$. Hence,

$$\begin{aligned}
\|f_0\|^p_{H^p} &\leqslant c \sum_k \sum_{l \in L(k)} \int_{\Omega_{kl} \setminus \bigcup_{m>l} \Omega_{km}} S_{\alpha,\mathscr{O}} f(x)^p\, dx \\
&\leqslant c \sum_k \int_{\{S_{\alpha,\mathscr{O}} f > t\} \cap \Omega_k} S_{\alpha,\mathscr{O}} f(x)^p\, dx \\
&= c \int_{\{S_{\alpha,\mathscr{O}} f > t\}} S_{\alpha,\mathscr{O}} f(x)^p\, dx,
\end{aligned}$$

and this would prove (3.3).

The argument needed to verify our claim is essentially the same as that in **[C-F2, J-T2]** to which we refer for more details.

The main step is to show that

$$\|b_k l_l\|_{L^2} \leqslant c \sup_{x \in \Omega_{kl} \setminus \Omega_{k,l+1}} S_{\alpha,\mathcal{O}} f(x) |\Omega_{kl}|^{1/2}. \tag{3.5}$$

To see this we pick $g \in L^2(\mathbf{R}^2)$ and put

$$\omega_m = \omega_{klm} = \left\{ M\chi_{\Omega_{kl}} > \tfrac{1}{100} \right\} \cap \Omega_{km} \setminus \Omega_{k,m+1}.$$

By Cauchy-Schwarz and the fact that

$$\sum_{m \leqslant l} \left| \left\{ x \in \omega_m : |x_i - y_i| < t_i,\ i = 1,2 \right\} \right| / t_1 t_2 \geqslant c$$

when $(y, t) \in B_{kl}^+$, we have

$$\begin{aligned}
\int b_{kl}(x) g(x)\, dx &= \int \int_{B_{kl}^+} b_{kl}(y,t) g(y,t)\, dy \frac{dt_1 dt_2}{t_1 t_2} \\
&\leqslant c \sum_{m \leqslant l} \int_{\omega_m} S_{\Omega_{km}} f(x) S_{\Omega_{km}} g(x)\, dx.
\end{aligned}$$

The way in which the sets Ω_{kl} were defined implies that this is less than

$$\begin{aligned}
c \int_{\bigcup_{m \leqslant l} \omega_m} S_{\alpha,\mathcal{O}} f(x) Sg(x)\, dx &\leqslant c \sup_{x \in \Omega_{kl} \setminus \Omega_{k,l+1}} S_{\alpha,\mathcal{O}} f(x) \left| \bigcup_{m \leqslant l} \omega_m \right|^{1/2} \|Sg\|_{L^2} \\
&\leqslant c \sup_{x \in \Omega_{kl} \setminus \Omega_{k,l+1}} S_{\alpha,\mathcal{O}} f(x) \left| \bigcup_{m \leqslant l} \omega_m \right|^{1/2} \|g\|_{L^2}.
\end{aligned}$$

Now (3.5) readily follows since $\bigcup_{m \leqslant l} \omega_m \subset \{ M\chi_{\Omega_{kl}} > \frac{1}{100} \} \cap \Omega_k \setminus \Omega_{k,l+1}$ and, hence, by the strong maximal theorem $|\bigcup_{m \leqslant l} \omega_m|^{1/2} \leqslant c |\Omega_{kl}|^{1/2}$.

There remains to prove (3.4). Proceeding as in the proof of (3.5) we find that

$$\langle g, f_1 \rangle \leqslant c \int_{S_{\alpha,\mathcal{O}} f \leqslant t} S_{\alpha,\mathcal{O}} f(x) Sg(x)\, dx \leqslant ct \|g\|_{H^1}, \tag{3.6}$$

for each Schwartz function g such that the support of $\hat{g}$ has a positive distance to the coordinate axes. This means that f_1 defines a linear functional on H^1 and, by (1.1), $\|f_1\|_{\mathrm{BMO}} \leqslant ct$. This completes the proof of (3.4) and the theorem.

Suppose $f \in \mathscr{S}'/\alpha$ and let $\mathcal{O} = \mathcal{O}_f$ be the directed family of open sets constructed in the proof above.

COROLLARY 3.2. *Let* $0 < \alpha < 1$. *Then*

$$\|f\|_{H^p} \approx \|S_{\alpha,\mathcal{O}} f\|_{L^p}, \qquad 0 < p < +\infty, \tag{3.7}$$

and

$$\|f\|_{\mathrm{BMO}} \approx \|S_{\alpha,\mathcal{O}} f\|_{L^\infty}. \tag{3.8}$$

PROOF. By referring to (2.4) and (2.5) we take care of one way. To show the converse inequalities, we first consider (3.7) in the case $0 < p \leqslant 1$: The proof of (3.3) shows that if $S_{\alpha,\mathcal{O}} f \in L^p$, then f has an atomic decomposition into p-atoms. Hence, $\|f\|_{H^p} \leqslant c \|S_{\alpha,\mathcal{O}} f\|_{L^p}$. Formally this corresponds to $t = 0$ in (3.3).

The proof of (3.6) on the other hand shows that

$$\langle g, f\rangle \leqslant c\int S_{\alpha,\mathscr{O}}f(x)Sg(x)\,dx \leqslant c\|S_{\alpha,\mathscr{O}}f\|_{L^p}\|g\|_{L^{p'}},$$

$1 < p < +\infty$. This readily gives us (3.7) for $1 < p < +\infty$.

The inequality $\|f\|_{\mathrm{BMO}} \leqslant c\|S_{\alpha,\mathscr{O}}f\|_{L^\infty}$ follows in a similar way. At least formally it is obtained by replacing f_1 by f and by putting $t = \infty$ in (3.6).

Combining Theorem 3.1 and (3.7) we obtain the following characterization, due to K. C. Lin [L], of the real interpolation spaces between H^p and BMO:

COROLLARY 3.3. *Let* $0 < p_0 < +\infty$ *and* $0 < \theta < 1$. *Then* $(H^{p_0}, \mathrm{BMO})_{\theta p} = H^p/\mathscr{A}$, $1/p = (1-\theta)/p_0$, *with equivalent (quasi-) norms.*

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