

ON COMPLETENESS OF THE PRODUCTS OF HARMONIC FUNCTIONS

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ABSTRACT. Let L be a partial differential operator in R^n with constant coefficients. We prove that, under some assumption on L , the set of products of the elements of the null-space of L forms a complete set in $L^p(D)$, $p \geq 1$, where D is any bounded domain. In particular, the products of harmonic functions form a complete set in $L^p(D)$, $p \geq 1$.

Let D be a bounded domain, $u_{lm} = r^l Y_{lm}(\omega)$, $l \geq 0$, where $Y_{lm}(\omega)$ are the spherical harmonics, $f \in L^2(D)$, and $\int_D f(x) u_{l_1 m_1}(x) u_{l_2 m_2}(x) dx = 0$, $\forall l_1, l_2, |m_1| \leq l_1, |m_2| \leq l_2$. Then $f = 0$.

1. Introduction. Let $D \subset R^3$ be a bounded domain with the origin inside D . The functions $u_{lm} = r^l Y_{lm}(\omega)$, where Y_{lm} are the normalized spherical harmonics, $l = 0, 1, 2, \dots, -l \leq m \leq l$, $\int_{S^2} Y_{lm}(\omega) \bar{Y}_{l'm'}(\omega) d\omega = \delta_{ll'} \delta_{mm'}$, and S^2 is the unit sphere. The functions u_{lm} form a complete set of harmonic functions. The question we are concerned with is completeness of the system $u_{l_1 m_1} u_{l_2 m_2}$ in $L^2(D)$. The result is

THEOREM 1. *Let*

$$(1) \quad \int_D f(x) u_{l_1 m_1}(x) u_{l_2 m_2}(x) dx = 0, \quad l_1, l_2 = 0, 1, 2, \dots, |m_1| \leq l_1, |m_2| \leq l_2.$$

Then $f = 0$.

In §2 we prove Theorem 1 and in §3 a more general Theorem 2.

2. Proof. Let $B_R = \{x: |x| \leq R\}$, $B_R \supset D$. Extend f to B_R by setting $f = 0$ in $B_R \setminus D$. Write (1) as

$$(2) \quad \int_0^R dr r^{2+l_1+l_2} \int_{S^2} f(r, \omega) Y_{l_1 m_1}(\omega) Y_{l_2 m_2}(\omega) d\omega = 0.$$

One has [1]

$$(3) \quad Y_{l_1 m_1}(\omega) Y_{l_2 m_2}(\omega) = \sum_{l=|l_1-l_2|}^{l_1+l_2} \rho(l_1, l_2, l) (l_1 l_2 m_1 m_2 | lm) Y_{lm}(\omega),$$

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where $\langle l_1 l_2 m_1 m_2 | lm \rangle$ are the Clebsch-Gordon coefficients, and $\rho(l_1, l_2, l) = \{(2l_1 + 1)(2l_2 + 1)/(2l + 1)\}^{1/2} \langle l_1 l_2 00 | l0 \rangle$. It follows from (3) that

$$(4) \quad \int_{S^2} Y_{l_1 m_1}(\omega) Y_{l_2 m_2}(\omega) \bar{Y}_{l_3 m_3}(\omega) d\omega = \rho(l_1, l_2, l_3) \langle l_1 l_2 m_1 m_2 | l_3 m_3 \rangle.$$

It follows from (2) and (4) that

$$(5) \quad \int_0^R dr r^{2+l_1+l_2} \sum_{l=|l_1-l_2|}^{l_1+l_2} f_{lm}(r) \rho(l_1, l_2, l) \langle l_1 l_2 m_1 m_2 | lm \rangle = 0.$$

Multiply (5) by $\langle l_1 l_2 m_1 m_2 | LM \rangle$, sum over m_1 and m_2 , and use the formula [1]:

$$(6) \quad \sum_{\substack{m_1, m_2 \\ m_1+m_2=m}} \langle l_1 l_2 m_1 m_2 | lm \rangle \langle l_1 l_2 m_1 m_2 | l' m' \rangle = \delta_{ll'} \delta_{mm'}$$

to obtain

$$(7) \quad \int_0^R dr r^{2+l_1+l_2} f_{LM}(r) \rho(l_1, l_2, L) = 0.$$

where $f_{LM} = \int_{S^2} f(r, \omega) \bar{Y}_{LM}(\omega) d\omega$. The coefficient $\rho(l_1, l_2, L) = 0$ unless $l_1 + l_2 + L$ is even and $|l_1 - l_2| \leq L \leq l_1 + l_2$. Take $l_2 = l_1 + L$. Then $\rho(l_1, l_1 + L, L) \neq 0$. Thus (7) yields

$$(8) \quad \int_0^R dr r^{2+2l_1+L} f_{LM}(r) = 0, \quad l_1 = 0, 1, 2, \dots$$

From (8) and Müntz's theorem [2] it follows that $f_{LM}(r) = 0$. Thus $f = 0$.

It is probably true that if $\mathcal{L}u_n = 0$, where \mathcal{L} is an elliptic operator of the second order in B_R and the system $\{u_n\}$ is a complete system of the solutions to the equation $Lu = 0$ in B_R , then the system $u_n(x)u_n'(x)$ is complete in $L^2(D)$, $D \subset B_R$. The system $\{u_n\}$ is a complete system of the solutions to the equation $Lu = 0$ in B_R if the closure of the linear span of u_n is the null space of L .

3. Generalizations. In this section we outline another approach to the problem. This approach is more general and leads to the following result. Let

$$Lu = \sum_{|\alpha| \leq l} a_\alpha D^\alpha u(x), \quad x \in R^n, n \geq 2,$$

where $a_\alpha = \text{const}$, α is a multi-index, $l > 1$. Let

$$M = \left\{ z: z \in C^n, \sum_{|\alpha| \leq l} a_\alpha z^\alpha = 0 \right\}.$$

ASSUMPTION A. *There exist two points a and b , $a, b \in M$, such that the tangent planes T_a and T_b to M at the points a and b are not parallel.*

Let $N(L) = \{u: Lu = 0\}$, $B_R = \{x: |x| \leq R, x \in R^n\}$, and $B_\epsilon(a) = \{z: |z - a| \leq \epsilon, z \in C^n\}$.

THEOREM 2. *Suppose that Assumption A holds, $f \in L^p(B_R)$, $p \geq 1$, $R > 0$ is an arbitrary fixed number, and*

$$\int fu_j u_{j'} dx = 0, \quad \forall u_j, u_{j'} \in N(L), \int = \int_{B_R}.$$

Then $f = 0$.

REMARK 1. The conclusion remains valid if u_j, u_j' run through a linear subset of $N(L)$ which is dense in $N(L)$ in L^p norm.

EXAMPLE. If $Lu = \Delta u, x \in R^3$, then

$$M = \left\{ z: \sum_{j=1}^3 z_j^2 = 0 \right\}.$$

Clearly Assumption A holds, e.g., for $a = (1, 0, i), b = (0, 1, i)$. Linear combinations of the functions $r^m Y_m, m \geq 0$, are dense in $N(\Delta)$ in $L^p(B_R)$. Therefore, Theorem 2 and Remark 1 imply the conclusion of Theorem 1.

REMARK 2. Note that the operator L in Theorem 2 can be elliptic, hyperbolic, parabolic, or neither. The proof of Theorem 2 is based on the following lemma:

LEMMA 1. Let M be a differentiable manifold in C^n of dimension $\tau = n - 1$, and Assumption A holds. Then, for any $\epsilon > 0$, the set $\{x + y\}, x \in M \cap B_\epsilon(a), y \in M \cap B_\epsilon(b)$, contains a ball $B_\delta(a + b) \subset C^n$, where $\delta = \delta(\epsilon) > 0$.

PROOF OF LEMMA 1. Consider the mapping $f: C^n \times C^n \rightarrow C^n$ given by the formula $f(x, y) = x + y$. This mapping is linear in x and y . Therefore its differential coincides with the mapping. The restriction of df on $M \times M$ is defined, in particular, on $T_a \times T_b$. If T_a is not parallel to T_b then the set $\{x + y\}$ contains a ball $B_\delta(a + b)$ if x runs through $T_a \cap B_\epsilon(a)$ and y runs through $T_b \cap B_\epsilon(b)$. For sufficiently small $\epsilon > 0$ the elements of $T_a \cap B_\epsilon(a)$ differ very little from the elements of $M \cap B_\epsilon(a)$. Therefore the conclusion of Lemma 1 holds.

REMARK 3. If $\dim M_1 = \tau_1, \dim M_2 = \tau_2$, and $\text{rank}(T_{a_1}, T_{a_2}) = n$, then the set $\{x + y\}$, where $x \in M_1 \cap B_\epsilon(a_1)$ and $y \in M_2 \cap B_\epsilon(a_2)$, contains a ball $B_\delta(a_1 + a_2), \delta = \delta(\epsilon) > 0$. Here (T_{a_1}, T_{a_2}) is the union of the systems of the basis vectors in T_{a_1} and T_{a_2} .

PROOF OF THEOREM 2. If

$$z \in M = \left\{ z: \sum_{|\alpha| \leq l} a_\alpha z^\alpha = 0 \right\},$$

then $\exp(x \cdot z) \in N(L)$, where

$$x \cdot z = \sum_{j=1}^n x_j z_j.$$

Consider $\exp\{x(z + v)\}$, where $z, v \in M$. By Lemma 1, the set $\{z + v\} \supset B_\delta(a + b)$, where a and b belong to M and T_a is not parallel to $T(b)$. Assume that

$$\int f(x) \exp\{x(z + v)\} dx = 0, \quad \forall z, v \in M.$$

Then

$$F(p) \stackrel{\Delta}{=} \int f(x) \exp(p \cdot x) dx = 0, \quad \forall p \in B_\delta(a + b).$$

Since $F(p)$ is an entire function of p , we conclude that $F(p) \equiv 0$ and $f(x) = 0$.

REMARK 4. Our argument shows that the assumption $f \in L^p(B_R)$, $p \geq 1$, can be relaxed: $f \in S'$, $\text{supp} f \subset B_R$ is sufficient, where S is the set of $C_0^\infty(R^n)$ test functions, and S' is the corresponding space of distributions.

REMARK 5. If

$$\sum_{|\alpha| \leq l} a_\alpha z^\alpha = \left(\sum_{j=1}^n c_j z_j \right)^l,$$

then Assumption A does not hold.

REMARK 6. It is interesting to prove Theorem 2 in the case when $a_\alpha = a_\alpha(x)$. This is an *open problem*.

REMARK 7. The results of this paper are useful in the study of multidimensional inverse problems [3, Chapter 6].

The idea that the set $\{z + v\}$ in the proof of Theorem 2 should contain an open set was suggested by Professor H. S. Shapiro with whom the author discussed the problem. The author is grateful to Professor H. S. Shapiro for the valuable suggestion and discussion, and to ONR for support.

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