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A COMPLEX SPACE WHOSE SPECTRUM IS NOT LOCALLY COMPACT ANYWHERE

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ABSTRACT. An example of a two-dimensional complex space is given with the property that the continuous spectrum of the global holomorphic functions is not locally compact at any point.

Introduction. The spectrum $S_c(\mathcal{O}(X))$ of the global holomorphic functions $\mathcal{O}(X)$ on a complex space $(X, \mathcal{O})$ is the set of all continuous complex-valued algebra homomorphisms on $\mathcal{O}(X)$ endowed with the Gelfand topology. This functional analytic concept has important applications to fundamental problems in complex analysis. For instance, due to a theorem of Igusa-Remmert-Iwahashi-Forster (see [**3**, 1.5]), if there is at least one point where the spectrum $S_c(\mathcal{O}(X))$ is not locally compact, then the complex space $(X, \mathcal{O})$ does not have a Stein envelope of holomorphy.[1]

In all the examples known of complex spaces $(X, \mathcal{O})$ without a Stein envelope of holomorphy, the set of points in the spectrum $S_c(\mathcal{O}(X))$ where local compactness fails is a nonempty, extremely small, proper subset of $S_c(\mathcal{O}(X))$; frequently, it consists of just one point [**2**, 5.3; **7**, 4.2, 4.3]. A natural question is whether such pathological points can be described as a thin subset of the spectrum. However, the purpose of this note is to show that the spectrum $S_c(\mathcal{O}(X))$ of a complex space $(X, \mathcal{O})$ need not be locally compact at any point at all; in the example constructed here, X is two dimensional. A complex space always refers to a reduced complex space with countable topology.

Construction. The example will be constructed in two steps. The main idea of the first step is to find a two-dimensional complex space $(X, \mathcal{O})$ whose spectrum $S_c(\mathcal{O}(X))$ has the following description. Take a two-dimensional complex plane and attach infinitely many two-dimensional complex planes transversally along every line of a countable dense subset of lines in the first plane. In the second step, the unique position of the initial plane will be eliminated by carrying out the construction of the first step for every plane attached to the initial plane.

First step of the construction. For every nonnegative integer $n \in \mathbf{N}$, set

$$S_n := \{(x, y) \in \mathbf{C}^2 : |x| \leq n+1, n \leq |y| \leq n + \tfrac{1}{2}\}.$$

Let D be the Reinhardt domain in $\mathbf{C}^2$ obtained by removing all the sets S_n, $n \in \mathbf{N}$, from $\mathbf{C}^2$.

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[1] Whether the converse of this statement holds is open.

Take countably many copies D_n, $n \in \mathbf{N}$, of D. Fix D_0, and for $n \geq 1$ consider each D_n as being transversal to D_0 in the following way. Choose a dense sequence $(\alpha_n)_{n\in\mathbf{N}}$ in $\mathbf{C}$ with $|\alpha_n| < n+1$ for $n \in \mathbf{N}$. For $\mathbf{N}^* := \mathbf{N}\backslash\{0\}$ let $\varphi\colon \mathbf{N}^* \to \mathbf{N}$ be a surjective map where each value is assumed countably many times and $\varphi(n) < n$ holds for every $n \in \mathbf{N}^*$. The existence of such a map can be seen by taking for every $n \in \mathbf{N}^*$ a $p \in \mathbf{N}$ with $p^2 \leq n < (p+1)^2$ and then defining $\varphi(n) := n - p^2$. In $\mathbf{C}^3$ with coordinates x, y, z, set $D_0 := D \times \{0\}$. Let D_m be the Reinhardt domain in $\{\alpha_{\varphi(m)}\} \times \mathbf{C}^2$ which remains after removing for every $n \in \mathbf{N}$ the set $\{\alpha_{\varphi(m)}\} \times S'_n$, where

$$S'_n := \{(y,z) \in \mathbf{C}^2 \colon n \leq |y| \leq n + \tfrac{1}{2}, |z| \leq n+1\}.$$

Roughly speaking, the desired complex space X is defined by attaching each D_n for $n \geq 1$ to D_0 along

$$R_n := \{(x,y,0) \in \mathbf{C}^3 \colon x = \alpha_{\varphi(n)}, n + \tfrac{1}{2} < |y| < n+1\}.$$

More precisely, observe that, for $n \in \mathbf{N}^*$, R_n is always an analytic subset of D_n; since $|\alpha_{\varphi(n)}| < n+1$ is valid, R_n is also an analytic subset of D_0. For every $n \in \mathbf{N}^*$, a complex space X_n having R_{n+1} as an analytic subset will be defined by induction.

Let X_1 be the complex space obtained by attaching D_1 to D_0 along R_1. This space

$$X_1 := D_0 +_{R_1} D_1$$

is the fiber sum (pushout) of D_0 and D_1 under the inclusion $R_1 \to D_0$ and $R_1 \to D_1$ [**6**]. Define

$$X_n := X_{n-1} +_{R_n} D_n$$

as the fiber sum of X_{n-1} and D_n under the inclusions $R_n \to X_{n-1}$ and $R_n \to D_n$.

For $n \in \mathbf{N}^*$ let $\iota_n\colon X_n \to X_{n+1}$ be the inclusion; ι_n embeds X_n in X_{n+1} as a closed subspace. Denote by X the direct limit of the expanding system (X_n, ι_n), i.e.

$$X := \varinjlim X_n.$$

With the direct limit topology, X is a Hausdorff space [**5**, 4.1]. To see that X can be given a complex structure, note that there is an open covering $\{U_n \colon n \in \mathbf{N}^*\}$ of X with the subsequent properties: For each $n \in \mathbf{N}^*$, U_n is a subset of X_n, and the complex charts on X defined by the U_n can be glued together to form a complex structure $\mathcal{O}$ on X [**4**, VA7].

In order to calculate the spectrum of the algebra $A := \mathcal{O}(X)$, take countably many copies $\mathbf{C}^2_n$, $n \in \mathbf{N}$, of $\mathbf{C}^2$ such that the first copy $\mathbf{C}^2_0 := \mathbf{C}^2 \times \{0\}$ is fixed and the other copies $\mathbf{C}^2_n := \{\alpha_{\varphi(n)}\} \times \mathbf{C}^2$, $n \geq 1$, are transversal to the first copy. Let $Y := \coprod_{n\in\mathbf{N}} \mathbf{C}^2_n$ denote their disjoint union with the natural complex structure. A is topologically isomorphic to

$$\left\{ (f_n)_{n\in\mathbf{N}} \in \prod_{n=0}^{\infty} \mathcal{O}(\mathbf{C}^2_n) \colon f_n\big|_{\{z=0\}} = f_0\big|_{\{x=\alpha_{\varphi(n)}\}}, n \geq 1 \right\}$$

and can therefore be viewed as a function algebra on Y.

LEMMA 1. *Let χ: $Y \to S_c(A)$ be the canonical map assigning to every point $y \in Y$ the point evaluation $\hat{y}$ defined by $\hat{y}(f) := f(y)$ for $f \in A$. The spectrum $S_c(A)$ is not locally compact at $\hat{y}$ for any $y \in \mathbf{C}_0^2$.*

PROOF. Let Q denote the quotient of Y by the equivalence relation R_χ associated to the map χ. Q is obtained by identifying the complex line $\{x = \alpha_{\varphi(n)}\}$ in $\mathbf{C}_0^2$ with the complex line $\{z = 0\}$ in $\mathbf{C}_n^2$ for every $n \geq 1$.

In $\mathbf{C}_0^2$, along each line $\{x = \alpha_{\varphi(n)}\}$ countably many planes are attached, because there are countably many preimages of $\varphi(n)$. Consequently, $\mathbf{C}_0^2$ is not locally compact along the line $\{x = \alpha_{\varphi(n)}\}$ for every $n \geq 1$. Since the sequence $(\alpha_n)_{n \in \mathbf{N}}$ is dense in $\mathbf{C}$, the plane $\mathbf{C}_0^2$ is not locally compact along this line. Such lines are dense in $\mathbf{C}^2_{\varphi_1(n)}$, implying that $\mathbf{C}^2_{\varphi_1(n)}$ is not locally compact anywhere. If $p: Y \to Q$ is the projection, then Q is not locally compact at $p(y)$ for any $y \in \mathbf{C}_0^2$.

Let $\bar{\chi}$: $Q \to S_c(A)$ be the unique map making the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\chi} & S_c(A) \\ & {}_p\searrow \quad \nearrow_{\bar{\chi}} & \\ & Q := Y/R_\chi & \end{array}$$

commutative. Since $\bar{\chi}$ is continuous, the assertion of Lemma 1 follows if $\bar{\chi}$ is proper, i.e. inverse images of compact sets are compact.

To prove that the map $\bar{\chi}$ really is proper, the fact that it is surjective will be used. This, in turn, can be seen as follows. A is the strongly dense inverse limit of the Fréchet algebras

$$A_m := \left\{ (f_0, \ldots, f_m) \in \prod_{n=0}^{m} \mathcal{O}(\mathbf{C}_n^2) : f_n\,|_{\{z=0\}} = f_0|_{\{x=\alpha_{\varphi(n)}\}},\ 1 \leq n \leq m \right\}$$

for $m \in \mathbf{N}$ with respect to the surjective maps

$$\pi_m : A_{m+1} \to A_m, \qquad (f_0, \ldots, f_{m+1}) \mapsto (f_0, \ldots, f_m).$$

According to a theorem of Arens [**1**, 5.21], the surjectivity of the projection $\sigma_m : A \to A_m$ onto the first $m+1$ components implies that, in the category of sets, $S_c(A)$ is the direct limit of the system $(S_c(A_m), \pi'_m)$ where $\pi'_m : S_c(A_m) \to S_c(A_{m+1})$ is the transposition $\psi \mapsto \psi \circ \pi_m$. The spectrum $S_c(A_m)$ is homeomorphic to the Stein space X'_m which is obtained from the disjoint union $\coprod_{n=0}^{m} \mathbf{C}_n^2$ when the line $\{x = \alpha_{\varphi(n)}\}$ in $\mathbf{C}_0^2$ is identified with the line $\{z = 0\}$ in $\mathbf{C}_n^2$ for $1 \leq n \leq m$ [**3**, 1.5]. Thus, $\bar{\chi}$ is surjective.

Now it can be shown that $\bar{\chi}$ is proper. Let K be a compact subset of $S_c(A)$. There is an $m \in \mathbf{N}$ with

$$(*) \qquad K \subset \sigma'_m(S_c(A_m)),$$

where $\sigma'_m : S_c(A_m) \to S_c(A)$ is the transportation. If this were not the case, there would be a sequence $(x_m)_{m \in \mathbf{N}}$ in K such that, when considered as a sequence in Q via the bijective map $\bar{\chi}$, the point x_m would lie in $\mathbf{C}_m^2$ but not on the line $\{z = 0\}$. Cartan's Theorem A would then ensure the existence of a fucntion $f_m \in \mathcal{O}(\mathbf{C}_m^2)$ with

$$f_m(x_m) = m \quad \text{and} \quad f_m = 0 \quad \text{on } \{z = 0\}$$

for every $m \geq 1$. Setting $f_0 := 0$, an element $f := (f_n)_{n\in\mathbf{N}}$ of A would exist which is unbounded on the sequence $(x_m)_{m\in\mathbf{N}}$. This is a contradiction, since f can be identified with its Gelfand transform which is bounded in K. Consequently, an $m \in \mathbf{N}$ exists with the property $(*)$. Since σ'_m is a topological embedding, K can be considered as a compact subset of X'_m and hence of Q, proving that $\bar{\chi}$ is proper. This completes the proof of Lemma 1.

Second step of the construction. Let $(\alpha_n)_{n\in\mathbf{N}}$, D and $D_0 := D \times \{0\}$ be as in the first step of the construction. Countably many copies D_n, $n \geq 1$, of D which are either parallel or transverse to D_0 will be defined and attached together by means of a surjective map $\varphi\colon \mathbf{N} \to \mathbf{N} \times \mathbf{N}$, $\varphi(0) = (0,0)$, $n \mapsto (\varphi_1(n), \varphi_2(n))$, where each value is assumed countably often and $\varphi_i(n) < n$ is true for every $n \in \mathbf{N}^*$ and for $i = 1,2$. Such a map is given, for example, by composing the map $\mathbf{N}^* \to \mathbf{N}$ used in the first step with a bijective map $\psi\colon \mathbf{N} \to \mathbf{N} \times \mathbf{N}$, $n \mapsto (\psi_1(n), \psi_2(n))$, satisfying $\psi_i(n) \leq n$ for $n \in \mathbf{N}$ and $i = 1,2$.

For $n \geq 1$, D_n will be defined by induction to be transverse to $D_{\varphi_1(n)}$. Let D_1 be as in the first step of the construction with $\alpha_0 := 0$. Call that part of D_1 in the yz-plane D', i.e. $D_1 = \{0\} \times D'$. If D_m has been defined to be parallel or transverse to D_0 and transverse to $D_{\varphi_1(m)}$ for $1 \leq m \leq n-1$, then $D_{\varphi_1(n)}$ is either parallel or transverse to D_0, since $\varphi_1(n) \leq n-1$. In the first case define $D_n := \{\alpha_{\varphi_2(n)}\} \times D'$ and in the second case set $D_n := D \times \{\alpha_{\varphi_2(n)}\}$.

The idea now is to obtain a complex space X by identifying each D_n with $D_{\varphi_1(n)}$ along

$$R_n := \left\{(x,y,z) \in \mathbf{C}^3 \colon x = \alpha_{\varphi_2(n)}, n + \tfrac{1}{2} < |y| < n+1, z = \alpha_{\varphi_2(\varphi_1(n))}\right\},$$

if $D_{\varphi_1(n)}$ is parallel to D_0 or along

$$R_n := \left\{(x,y,z) \in \mathbf{C}^3 \colon x = \alpha_{\varphi_2(\varphi_1(n))}, n + \tfrac{1}{2} < |y| < n+1, z = \alpha_{\varphi_2(n)}\right\},$$

if $D_{\varphi_1(n)}$ is transverse to D_0. Because $|\alpha_{\varphi_2(\varphi_1(n))}| < n+1$ and $|\alpha_{\varphi_2(n)}| < n+1$ hold, R_n is an analytic subset of D_n as well as of $D_{\varphi_1(n)}$, and such an identification is possible [**6**].

To be more exact, X will again denote the direct limit of expanding system of complex spaces X_n, $n \geq 1$, defined by induction as follows:

$$X_1 := D_0 +_{R_1} D_1, \qquad X_n := X_{n-1} +_{R_n} D_n.$$

As before, X is a complex space [**5**, **4**].

The spectrum of $A := \mathcal{O}(X)$ is determined by considering in $\mathbf{C}^3$ countably many copies $\mathbf{C}^2_n$, $n \in \mathbf{N}$, of $\mathbf{C}^2$ parallel to $\mathbf{C}^2 \times \{0\}$ or to $\{0\} \times \mathbf{C}^2$ which are defined by induction. Let $\mathbf{C}^2_0 := \mathbf{C}^2 \times \{0\}$ and $\mathbf{C}^2_1 := \{0\} \times \mathbf{C}^2$. If $\mathbf{C}^2_m$ has already been defined for $1 \leq m \leq n-1$, then $\mathbf{C}^2_{\varphi_1(n)}$ is either parallel or transverse to $\mathbf{C}^2_0$. If the former is true, put $\mathbf{C}^2_n := \{\alpha_{\varphi_2(n)}\} \times \mathbf{C}^2$ and otherwise define $\mathbf{C}^2_n := \mathbf{C}^2 \times \{\alpha_{\varphi_2(n)}\}$. A is topologically isomorphic to the set of elements $(f_n)_{n\in\mathbf{N}} \in \prod_{n=0}^{\infty} \mathcal{O}(\mathbf{C}^2_n)$ satisfying the following conditions for all $n \geq 1$:

$$f_n \Big|_{\{z=\alpha_{\varphi_2(\varphi_1(n))}\}} = f_{\varphi_1(n)}\Big|_{\{x=\alpha_{\varphi_2(n)}\}}$$

when $\mathbf{C}^2_n$ is transverse to $\mathbf{C}^2_0$ and otherwise

$$f_n \Big|_{\{x=\alpha_{\varphi_2(\varphi_1(n))}\}} = f_{\varphi_1(n)}\Big|_{\{z=\alpha_{\varphi_2(n)}\}}.$$

LEMMA 2. *The spectrum $S_c(A)$ is not locally compact anywhere.*

PROOF. Denote by $Y := \coprod_{n \in \mathbf{N}} \mathbf{C}_n^2$ the disjoint union of the planes $\mathbf{C}_n^2$ with the natural complex structure; A is a subalgebra of $\mathcal{O}(Y)$. Let $\psi\colon Y \to S_c(A)$ be the map $y \mapsto \hat{y}$ for $\hat{y}(f) := f(y)$. The quotient Q of Y by the equivalence relation R_χ given by χ is not locally compact at any point. To verify this, notice that Q is obtained from Y by identifying the line

$$\{z = \alpha_{\varphi_2(\varphi_1(n))}\} \quad \text{resp.} \quad \{x = \alpha_{\varphi_2(\varphi_1(n))}\}$$

in $\mathbf{C}_n^2$ with the line

$$\{x = \alpha_{\varphi_2(n)}\} \quad \text{resp.} \quad \{z = \alpha_{\varphi_2(n)}\}$$

in $\mathbf{C}^2_{\varphi_1(n)}$ for every $n \geq 1$; the choice of the line depends upon whether $\mathbf{C}^2_{\varphi_1(n)}$ is parallel or transverse to $\mathbf{C}_0^2$. Since there are countably many preimages of $\varphi(n)$, there are countably many planes attached to $\mathbf{C}^2_{\varphi_1(n)}$ along the line given by $x = \alpha_{\varphi_2(n)}$ resp. $z = \alpha_{\varphi_2(n)}$. Hence, $\mathbf{C}^2_{\varphi_1(n)}$ is not locally compact along this line. Such lines are dense in $C^2_{\varphi_1(n)}$, implying the $C^2_{\varphi_1(n)}$ is not locally compact anywhere. Because φ is surjective, no plane $\mathbf{C}_n^2$ has a point at which it is locally compact. Thus, Q is not locally compact anywhere.

Let $\bar{\chi}\colon Q \to S_c(A)$ be the canonical map induced by χ. As in Lemma 1, it follows from a theorem of Arens [**1**, 5.21] that the continuous injective map $\bar{\chi}$ is surjective. Together with Cartan's Theorem A, this implies that $\bar{\chi}$ is proper, proving Lemma 2.

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