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A GENERALIZATION OF THE LIGHTBULB THEOREM AND PL I-EQUIVALENCE OF LINKS

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ABSTRACT. By the "lightbulb theorem" I mean the result that a knot of S^1 in $S^1 \times S^2$ which meets some S^2 factor in a single transverse point is isotopic to an S^1 factor. We prove an analogous result for knots of S^n in $S^n \times S^2$, and apply it to answer a question of Rolfsen concerning PL I-equivalence of links.

Introduction. In [9], Rolfsen asked the following

QUESTION. Do there exist links $L = L_1 \cup \cdots \cup L_\mu$ and $L' = L'_1 \cup \cdots \cup L'_\mu$ of n-spheres in an $(n+2)$-manifold M such that L and L' are I-equivalent and, for each $i = 1, \ldots, \mu$, L_i and L'_i are concordant knots, yet L fails to be concordant to L'?

(This refers to the PL category; I-equivalence is the relation that results when concordances are not required to be locally flat.) The question arises because Theorem 3 of [9] asserts that there are no such links in S^{n+2}. The proof of that theorem shows that the answer is no if n is even (since the knot concordance group is zero in even dimensions). We shall show that there are examples for every odd n. The case $n = 1$ is easily described. We take $M = S^1 \times S^2$. Let x and y be two points of S^2, set $L_1 = L'_1 = S^1 \times \{x\}$ and $L_2 = S^1 \times \{y\}$, and let L'_2 be the result of locally tying a trefoil in L_2. Then $L = L_1 \cup L_2$ and $L' = L'_1 \cup L'_2$ satisfy the desired conditions. In fact L_2 and L'_2 are ambient isotopic; this is a special case of what is sometimes known as the lightbulb theorem.

To construct examples for greater values of n, we would like to replace S^1 by S^n throughout (and the trefoil by some nonslice knot of S^n). To see that this produces links with the right properties, we prove in §1 a higher-dimensional version of the lightbulb theorem. For this it seems to be necessary to work in the smooth category. A little triangulation theory gives results for PL case, and hence our examples, in §2.

For us, a knot of M in N will mean a submanifold of N isomorphic (i.e., diffeomorphic or PL homeomorphic, as appropriate) to M, rather than an embedding of M in N. (In the PL case, the submanifold is to be locally flat.) All manifolds will be oriented, and all isomorphisms between manifolds will be orientation preserving. If M_1 and M_2 are submanifolds of N and $h\colon N \to N$ is an isomorphism, the statement $h(M_1) = M_2$ means that $h(M_1)$ and M_2 are equal as oriented manifolds, i.e., that $h \mid M_1\colon M_1 \to M_2$ is also orientation preserving.

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1. The smooth case. In this section we work in the smooth category. Our aim is to prove

THEOREM 1. *Let Σ be a smooth knot of S^n in $S^n \times S^2$ such that Σ meets some $\{z\} \times S^2$ in a single transverse point, and let $\Sigma_0 = S^n \times \{x\}$ for some $x \in S^2$. For $n \geqslant 3$, the following are equivalent*:

(a) Σ *is (ambient) isotopic to* Σ_0;

(b) Σ *is ambiently concordant to* Σ_0;

(c) Σ *is concordant to* Σ_0;

(d) *both of the following hold*:

  (i) Σ *is homotopic to* Σ_0;

  (ii) $i(\Sigma) \subset S^{n+3}$ *is unknotted, where* i: $S^n \times S^2 \hookrightarrow S^{n+3}$ *is the standard inclusion.*

If $n = 2$ *then* (b), (c), *and* (d) *are equivalent.*

REMARK 1. By *ambiently concordant* we mean that there is a diffeomorphism h: $S^n \times S^2 \to S^n \times S^2$ such that $h(\Sigma_0) = \Sigma$ and h is concordant to the identity.

REMARK 2. The "lightbulb theorem" is the statement that if $n = 1$ then (a) always holds (provided that Σ is correctly oriented). For $n > 1$ this is not the case. If we take Σ to be the graph of an essential map $S^n \to S^2$ then (d)(i) will not hold. That (d)(ii) may also fail is shown by examples of the kind we need in §2, as we now describe.

Let K be a knot of S^n in S^{n+2}. With Σ_0 as above, define $\Sigma_K = \Sigma_0 \# K$. Then Σ_K is homotopic to Σ_0, but $i(\Sigma_K)$ is unknotted in S^{n+3} iff $j(K)$ is, where j: $S^{n+2} \to S^{n+3}$ is the standard inclusion. Let $C^{m,n}$ denote the group of concordance classes of knots of S^n in S^m. According to [**4**, Theorem 1.2], $C^{m,n}$ is identical with the group of isotopy classes of such knots for $m \geqslant n + 3$, so $j(K)$ is unknotted iff K represents an element of the kernel of j_*: $C^{n+2,n} \to C^{n+3,n}$.

PROPOSITION 1. *With the notation above*

$$\operatorname{Im} j_* \cong \begin{cases} 0, & \text{if } n \equiv 0 \bmod 2, \\ 0 \text{ or } \mathbf{Z}/2, & \text{if } n \equiv 1 \bmod 4, \\ \mathbf{Z}, & \text{if } n \equiv 3 \bmod 4, \end{cases}$$

and Ker j_* *is not finitely generated if n is odd.*

PROOF. Consider first Im j_*. For $n \equiv 0 \bmod 2$ (respectively $n = 1$) there is nothing to prove, since $C^{n+2,n} = 0$ (respectively $C^{4,1} = 0$). Suppose that n is odd and $n \geqslant 5$. In [7], Levine studies the group $\Theta^{m,n}$ of concordance classes of knots of homotopy n-spheres in S^m; this contains $C^{m,n}$ as a subgroup. According to [**4**] (see the remark preceding the statement of Corollary 6.6), the image of j_* is the intersection of $C^{n+3,n}$ with the kernel of the homomorphism $\omega_3(n,3)$: $\Theta^{n+3,n} \to \pi_n(G_3, SO_3)$ appearing in Levine's exact sequence $(3)_3$; Levine calls this group $\Sigma_0^{n+3,n}$. The result now follows from Theorem 6.7 of [**7**].

There remains only the case $n = 3$. This is proved in [4, Theorem 5.17] with $C^{5,3}$ and $C^{6,3}$ replaced by the groups of concordance classes of embeddings of S^3 in S^5 and S^6, respectively. However, since $\Gamma_4 = 0$ (Cerf [1]), we obtain the same group from embeddings as from submanifolds.

For the claim about $\operatorname{Ker} j_*$ we need to know that $C^{n+2,n}$ is not finitely generated for odd n. The knot cobordism groups determined by Levine in [8] coincide with $C^{n+2,n}$ for $n = 1$ and $n = 3$, and with $\Theta^{n+2,n}$ for other odd n. In the latter cases, $C^{n+2,n}$ has finite index in $\Theta^{n+2,n}$ since the group of homotopy n-spheres is finite (Kervaire and Milnor [6]). □

This shows that (d)(ii) need not be satisfied, at least when $n \equiv 3 \bmod 4$.

LEMMA 1. *Let Σ and Σ_0 be as in Theorem 1. Assume that Σ and Σ_0 are homologous. Then there is a diffeomorphism $f\colon S^n \times D^3 \to S^n \times D^3$ such that $f(\Sigma_0) = \Sigma$.*

PROOF. Let $S^n = D^n_- \cup_\partial D^n_+$, where $z \in \operatorname{int} D^n_-$. We may assume that $\Sigma \cap (D^n_- \times S^2) = \Sigma_0 \cap (D^n_- \times S^2) = D^n_- \times \{x\}$. Let $\Delta = \Sigma \cap (D^n_+ \times S^2)$ and $\Delta_0 = \Sigma_0 \cap (D^n_+ \times S^2) = D^n_+ \times \{x\}$; these are properly embedded n-discs in $D^n_+ \times S^2$ with $\partial\Delta = \partial\Delta_0$. We can also regard them as being contained in $S^{n+2} = D^n_+ \times S^2 \cup_\partial S^{n-1} \times D^3$. We claim that there exists a diffeomorphism $g\colon S^{n+2} \to S^{n+2}$ such that

(1) $g \mid S^{n-1} \times D^3$ is equal to the identity;

(2) $g(\Delta_0) = \Delta$;

(3) g is isotopic to the identity.

Now g is defined on $S^{n-1} \times D^3$ by (1); we first extend it over Δ_0. There is a diffeomorphism $g_1\colon \Sigma_0 \to \Sigma$, and we may assume that its restriction to $D^n_- \times \{x\}$ is the identity since any two orientation-preserving embeddings of the n-disc $D^n_- \times \{x\}$ in Σ are isotopic. Extend g by $g_1 \mid \Delta_0$.

Next we claim that g extends over a tubular neighborhood T_0 of Δ_0. The product structure on $\partial(D^n_+ \times S^2)$ gives a normal framing of $\partial\Delta \subset \partial(D^n_+ \times S^2)$, and we are claiming that this extends to a normal framing of $\Delta \subset D^n_+ \times S^2$, or equivalently that Σ has trivial normal bundle in $S^n \times S^2$. For $n = 2$ this follows from the assumption that Σ and Σ_0 are homologous; for $n \neq 2$ any SO_2-bundle over S^n is trivial.

At this point, g is defined on $(S^{n-1} \times D^3) \cup T_0 \cong D^{n+2}$ and is therefore isotopic to the inclusion, allowing us to extend it to all of S^{n+2}.

Now regard $S^n \times D^3$ as D^{n+3} with an n-handle attached along $S^{n-1} \times D^3 \subset S^{n+2} = \partial D^{n+3}$. We obtain f by extending g over D^{n+3} using (3), and over the handle using (1). □

The main part of the proof of Theorem 1 is to show that the f provided by Lemma 1 can be chosen so that its restriction to $S^n \times S^2$ is concordant to the identity. Diffeomorphisms of $S^p \times S^q$ have been classified up to concordance by Sato [10], for a certain range of values of p and q. Unfortunately, this classification is not applicable when $q = 2$; we can however use Sato's methods. The following lemma is a consequence of Proposition 1.1 of [10]; we include a proof for the reader's convenience.

LEMMA 2. *Let* $f\colon D^{n+1}\times S^2\to D^{n+1}\times S^2$ *be a diffeomorphism inducing the identity on* $H_2(D^{n+1}\times S^2)$, *where* $n\geqslant 2$. *Then the restriction of* f *to* $S^n\times S^2$ *is concordant to the identity.*

PROOF. Let k be the inclusion of $0\times S^2$ into $D^{n+1}\times S^2$. Then k and $f\circ k$ are homotopic, and therefore isotopic by Haefliger [3, Théorème d'existence, (b)]; hence we may assume that $f|0\times S^2=k$. By the tubular neighborhood theorem we may further assume that f maps $\frac{1}{2}D^{n+1}\times S^2$ to itself by a bundle isomorphism. Since $\pi_2(SO_{n+1})=0$ we can arrange that $f|\frac{1}{2}D^{n+1}\times S^2$ is the identity. Identify $X=(D^{n+1}-\operatorname{int}(\frac{1}{2}D^{n+1}))\times S^2$ with $S^n\times S^2\times I$; then $f|X$ is the desired concordance. □

PROOF OF THEOREM 1. The implications (a) ⇒ (b) ⇒ (c) ⇒ (d)(i) are trivial. Also (c) implies that $i(\Sigma)$ is null-concordant, and hence (d)(ii) by Theorem 1.2 of [4]. Moreover, (b) ⇒ (a) for $n\geqslant 3$ by a theorem of Cerf [2, Corollary 1], so we need only show that (d) ⇒ (b) (for $n\geqslant 2$). Assume therefore that Σ and Σ_0 satisfy (d).

Let f be a diffeomorphism of $S^n\times D^3$ given by Lemma 1. Identify S^{n+3} with $S^n\times D^3\cup_\partial D^{n+1}\times S^2$ and let $i\colon S^n\times D^3\to S^{n+3}$ be the inclusion. (Thus $i|S^n\times S^2$ is the standard inclusion, previously denoted by i.) Since $if(S^n\times 0)$ is isotopic to $i(\Sigma)$, (d)(ii) implies that there is a diffeomorphism g_1 of S^{n+3}, isotopic to the identity, such that $g_1f(S^n\times 0)=S^n\times 0$. Let k be the diffeomorphism of $S^n\times 0$ obtained by restricting g_1f. Note that $f\circ(k^{-1}\times\mathrm{id})$ also satisfies the conclusion of Lemma 1; replacing f by this diffeomorphism we may assume that $g_1f|S^n\times 0$ is the identity. Let $h=f|S^n\times S^2$; the proof will be completed by showing that h is concordant to the identity.

By the tubular neighborhood theorem, there is a diffeomorphism g_2 of S^{n+3}, isotopic to the identity, such that g_2g_1f maps $S^n\times D^3$ to itself by a bundle isomorphism. Then g_2g_1 maps $D^{n+1}\times S^2$ to itself and induces the identity on $H_2(D^{n+1}\times S^2)$ because

$$\begin{aligned}\mathrm{Lk}\big(g_2g_1(0\times S^2),S^n\times 0\big)&=\mathrm{Lk}\big(g_2g_1(0\times S^2),g_2g_1f(S^n\times 0)\big)\\&=\mathrm{Lk}\big(0\times S^2,f(S^n\times 0)\big)\\&=\mathrm{Lk}(0\times S^2,\Sigma)\\&=\mathrm{Lk}\big(0\times S^2,\Sigma_0\big)\\&=\mathrm{Lk}(0\times S^2,S^n\times 0).\end{aligned}$$

By Lemma 2, $g_2g_1|S^n\times S^2$ is concordant to the identity. Denote by $C_0(S^n\times S^2)$ the group of concordance classes of those diffeomorphisms of $S^n\times S^2$ which induce the identity on $H_*(S^n\times S^2)$, and let $\alpha\colon\pi_n(SO_3)\to C_0(S^n\times S^2)$ be the evident homomorphism. We have shown that h represents an element of the image of α. Since $\pi_2(SO_3)=0$ we assume from now on that $n\geqslant 3$.

There is a homomorphism $\beta\colon C_0(S^n\times S^2)\to\pi_n(S^2)$ which sends the concordance class of a diffeomorphism g to the image of the homotopy class of $g(\Sigma_0)$ under the projection $\pi_n(S^n\times S^2)\to\pi_n(S^2)$; the condition (d)(i) says that $\beta([h])=0$.

We have a commutative diagram

$$\begin{array}{ccc} \pi_n(SO_3) & \xrightarrow{\alpha} & C_0(S^n \times S^2) \\ \uparrow p_* & & \downarrow \beta \\ \pi_n(S^3) & \xrightarrow{H_*} & \pi_n(S^2) \end{array}$$

Here p and H are the double covering $S^3 \to SO_3$ and the Hopf fibration $S^3 \to S^2$ respectively, and therefore induce isomorphisms on π_n. Hence h is concordant to the identity, as desired. □

2. The PL case and I-equivalence of links. In this section all manifolds, homeomorphisms, etc. will be PL unless otherwise stated; in particular, S^n will denote the PL n-sphere. All submanifolds will be locally flat. Let M^{n+2} be a closed manifold. We denote by $\mathscr{K}(M)$ ($\mathscr{A}\mathscr{C}(M), \mathscr{C}(M)$) the set of ambient isotopy (ambient concordance, concordance) classes of knots of S^n in M. Suppose that M_α is a smoothing of M (i.e., M_α is a smooth manifold obtained by giving M a smooth structure such that the identity $M \to M_\alpha$ is a piecewise-differentiable (PD) homeomorphism). By a smooth knot of S^n in M_α we mean a smooth submanifold K of M_α admitting a PD homeomorphism $S^n \to K$; the induced smooth structure on S^n is not required to be standard. We denote by $\mathscr{K}(M_\alpha)$ ($\mathscr{A}\mathscr{C}(M_\alpha), \mathscr{C}(M_\alpha)$) the set of smooth (ambient) isotopy (ambient concordance, concordance) classes of smooth knots of S^n in M_α.

Let K be a PL knot of S^n in M and K^* a smooth knot of S^n in M_α. We call K an *ambient triangulation* of K^*, and K^* an *ambient smoothing* of K, if there is a PD isotopy H_t: $M \to M_\alpha$ such that H_0 is the identity and $H_1(K) = K^*$. Any smooth knot has an ambient triangulation, and this induces functions $t_{\mathscr{X}}$: $\mathscr{X}(M_\alpha) \to \mathscr{X}(M)$ for $\mathscr{X}$ any one of $\mathscr{K}$, $\mathscr{A}\mathscr{C}$ or $\mathscr{C}$. (This follows from the refinements of Whitehead's triangulation theorems given in [**5**, Part I, §13].) By Theorem 2 of Wall [**11**], each $t_{\mathscr{X}}$ is surjective, and (because concordances between knots, being also of codimension 2, can be ambiently smoothed) $t_{\mathscr{C}}$ is an isomorphism. Denote the standard smoothing of S^n by S^n_{Diff}.

THEOREM 2. *Let Σ be a PL knot of S^n in $S^n \times S^2$ such that Σ meets some $\{z\} \times S^2$ in a single transverse point, and let $\Sigma_0 = S^n \times \{x\}$ for some $x \in S^2$. For $n \geqslant 3$, the following are equivalent*:

(a) *Σ is ambient isotopic to Σ_0;*

(b) *Σ is ambiently concordant to Σ_0;*

(c) *Σ is concordant to Σ_0;*

(d) *both of the following hold*:

(i) *Σ is homotopic in Σ_0;*

(ii) *$i(\Sigma^*) \subset S^{n+3}_{\mathrm{Diff}}$ is unknotted, where Σ^* is an ambient smoothing of Σ and i is the standard inclusion of $S^n_{\mathrm{Diff}} \times S^2_{\mathrm{Diff}}$ in S^{n+3}_{Diff}.*

If $n = 2$ then (b), (c), *and* (d) *are equivalent.*

PROOF. Since Σ can be smoothed by a PD isotopy which is arbitrarily close to the identity in the C^1-topology, we can take Σ^* to meet $\{z\} \times S^2_{\mathrm{Diff}}$ in a single transverse point. The result now follows from the commutative diagram

$$\begin{array}{ccccc} \mathscr{K}(S^n_{\mathrm{Diff}} \times S^2_{\mathrm{Diff}}) & \to & \mathscr{AC}(S^n_{\mathrm{Diff}} \times S^2_{\mathrm{Diff}}) & \to & \mathscr{C}(S^n_{\mathrm{Diff}} \times S^2_{\mathrm{Diff}}) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \mathscr{K}(S^n \times S^2) & \to & \mathscr{AC}(S^n \times S^2) & \to & \mathscr{C}(S^n \times S^2) \end{array}$$

and Theorem 1. □

Regard $\mathscr{C}(S^n \times S^2)$ as a pointed set with basepoint the class of Σ_0. Combining Theorem 2 and Proposition 1 we have

COROLLARY 1. *The kernel of the map $\mathscr{C}(S^{n+2}) \to \mathscr{C}(S^n \times S^2)$ given by connected sum with Σ_0 is not finitely generated if n is odd.* □

We can now answer Rolfsen's question.

THEOREM 3. *For each odd integer n there are two-component links $L = L_1 \cup L_2$ and $L' = L'_1 \cup L'_2$ of n-spheres in $S^n \times S^2$ such that L_i and L'_i are concordant knots for $i = 1, 2$ and L is I-equivalent, but not concordant, to L'.*

PROOF. For $n > 1$, let K be any knot representing a nontrivial element of the kernel of $\mathscr{C}(S^{n+2}) \to \mathscr{C}(S^n \times S^2)$; for $n = 1$ let K be any knot which is not algebraically slice. Let x and y be any two points of S^2. Let $L_1 = L'_1 = S^n \times \{x\}$, let $L_2 = S^n \times \{y\}$, and let $L'_2 = L_2 \# K$, so that L'_2 is concordant to L_2 by choice of K (or by the lightbulb theorem if $n = 1$). Then L and L' are I-equivalent; we need to show that they are not concordant. Suppose that $C = C_1 \cup C_2 \subset S^n \times S^2 \times I$ is a concordance between them. We can remove a neighborhood of C_1 and sew it back so as to obtain a manifold $W \supset C_2$ with $\partial(W, C_2) = (S^{n+2}, 0) \sqcup -(S^{n+2}, K)$. Then W is a homology cobordism so that K is algebraically slice, and hence slice if $n > 1$, contrary to assumption. (In fact, if $n > 1$ then W is an h-cobordism and therefore a product.) □

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