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A TYPICAL PROPERTY OF BAIRE 1 DARBOUX FUNCTIONS

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ABSTRACT. It is well known that a real-valued, bounded, Baire class one function of a real variable is the derivative of its indefinite integral at every point except possibly those in a set which is both of measure zero and of first category. In the present paper, a bounded, Darboux, Baire class one function is constructed to have the property that its indefinite integral fails to be differentiable at non-σ-porous set of points. Such functions are then shown to be "typical" in the sense of category in several standard function spaces.

We shall denote by $\mathcal{B}^1$, $\mathcal{DB}^1$, $b\mathcal{B}^1$, and $b\mathcal{DB}^1$ the spaces of Baire class 1 functions, Baire 1 Darboux functions, bounded Baire 1 functions, and bounded Baire 1 Darboux functions, respectively, all defined on the interval $[0,1]$ and all equipped with the topology of uniform convergence. The word "typical" in the title refers to any property which holds for most elements in a space of functions in the sense of category; i.e., the collection of functions not possessing the property is of first category in the space. Perhaps a better title for the present paper would be, "*Another typical property of $\mathcal{DB}^1$ functions*," for many such properties are known. A virtually complete catalog of such results can be found in the survey article [**2**] by Ceder and Pearson.

In the process of writing [**4**] the present authors became curious about the "size" of the set of points at which bounded functions in certain classes can fail to be the derivatives of their indefinite integrals. Clearly, for functions in $b\mathcal{B}^1$ this exception must be both of measure zero and of first category. In [**4**] we were especially concerned with finding circumstances under which this set would be σ-porous. (The concept of a σ-porous set was introduced in [**3**] by E. P. Dolzenko. The porosity of a set E of real numbers at the point x on the real line is the value

$$\limsup_{r\downarrow 0} \frac{l(x,r,E)}{r},$$

where $l(x,r,E)$ denotes the length of the largest open interval contained in the intersection of the complement of E with the interval $(x-r, x+r)$. The set E is porous if it has positive porosity at each of its points, and it is σ-porous if it is a countable union of porous sets. Thus, σ-porous sets are of both measure zero and first category. Dolzenko showed that σ-porous sets are the natural exceptional sets for certain types of boundary behavior for complex functions defined, for example, in the upper half plane. More recently, σ-porous sets have been found to play a useful role in describing behavior of real functions.) In [**4**] we showed that any function in the subclass of $b\mathcal{B}^1$ consisting of bounded approximately symmetric functions is the derivative of its indefinite integral except at a σ-porous set of points. As noted in [**4**], it is easy to see that an arbitrary function in $b\mathcal{B}^1$ need

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not have this property by considering the characteristic function of a non-σ-porous perfect set of measure zero. (Such perfect sets have been exhibited in [**5**, **6**, and **8**].) However, that example leaves at least a couple of things to be desired. Firstly, the function is equivalent (i.e., equal a.e.) to a function which *is* the derivative of its indefinite integral everywhere, and secondly, that example does not have the Darboux property, leaving one to wonder about the problem for functions in $b\mathcal{DB}^1$. The purpose of the present paper is to remedy this situation as follows: We shall first construct a function f in $b\mathcal{DB}^1$ for which the set of points x at which $\lim_{h\to 0}\frac{1}{h}\int_0^h f(x+t)\,dt$ fails to exist is not σ-porous, and then we shall show that such functions are typical, not only in $b\mathcal{DB}^1$, but in each of the spaces mentioned in the opening paragraph.

For notation we shall use $|S|$ to denote the Lebesgue measure of a measurable set S, $S\backslash T$ to denote the intersection of S with the complement of T, and χ_S to denote the characteristic function of S. Also, if $f\in\mathcal{B}^1$, we let $N(f)=\{x\colon \lim_{h\to 0}\frac{1}{h}\int_0^h f(x+t)\,dt$ does not exist$\}$ and, finally, we let $\mathcal{N}$ denote the set of all $f\in\mathcal{B}^1$ for which $N(f)$ is a σ-porous set.

THEOREM 1. *If E is any $\mathcal{F}_\sigma$ set of measure zero in $[0,1]$, there is a bounded Baire class one Darboux function $f\colon[0,1]\to\mathbf{R}$ such that*

(i) *$E=N(f)$, and*

(ii) *f is continuous at each $x\notin E$.*

PROOF. Let E be an $\mathcal{F}_\sigma$ set of measure zero in $[0,1]=I_0$. The set E can be expressed as $E=\bigcup_{n=1}^\infty F_n$, where the F_n's are closed and pairwise disjoint. As a first case, let us further suppose that each of the F_n's is a perfect set. We shall inductively define two seqences $\{H_n\}$ and $\{H_n^*\}$ of open sets and utilize these in constructing the function f.

First, enumerate the component intervals of $I_0\backslash F_1$ as a sequence $\{G_{1i}\colon i=1,2,\ldots\}$. Let N_1 be so large that $|I_0\cap(\bigcup_{i=1}^{N_1}G_{1i})|\geq 3|I_0|/4$. In general, if N_k has been defined, let N_{k+1} be large enough so that if I is any component of $I_0\backslash\bigcup_{i=1}^{N_k}G_{1i}$, then

$$\left|I\cap\left(\bigcup_{i=N_k+1}^{N_{k+1}}G_{1i}\right)\right|\geq\left(1-2^{-k-2}\right)|I|.$$

For notational purposes set $N_0=0$. For each $k=1,2,\ldots$, set

$$H_{1k}=\bigcup_{i=N_{2k-2}+1}^{N_{2k-1}}G_{1i}\quad\text{and}\quad H_{1k}^*=\bigcup_{i=N_{2k-1}+1}^{N_{2k}}G_{1i}.$$

Next, let $H_1=\bigcup_{k=1}^\infty H_{1k}$ and $H_1^*=\bigcup_{k=1}^\infty H_{1k}^*$. Then H_1 and H_1^* are disjoint open sets with $H_1\cup H_1^*=I_0\backslash F_1$. For each i, let $G_{1i}=(a_i,b_i)$ and define a function $f_1\colon I_0\to\mathbf{R}$ by

$$f_1(x)=\sum_{k=1}^\infty\sum_{i=1}^\infty\left[\chi_{H_{1k}}(x)-\chi_{H_{1k}^*}(x)\right]\cdot\alpha\left[\frac{x-a_i}{b_i-a_i}\right],$$

where $\alpha\colon\mathbf{R}\to\mathbf{R}$ is given by

$$\alpha(x)=\begin{cases}0, & x\notin[0,1],\\ 4x, & x\in[0,\frac{1}{2}],\\ 4(1-x), & x\in(\frac{1}{2},1].\end{cases}$$

Note that for each k only finitely many of the functions in the inner sum are not identically zero and that each is continuous. Consequently, f_1 is a Baire one function. The range of f_1 is $[-2,2]$, f_1 is continuous at each point of $I_0\backslash F_1$, and f_1 assumes every value in its range infinitely often in every neighborhood of each point of the perfect set F_1. Consequently, f_1 is a Darboux function. Furthermore, note that if I is any component of $I_0\backslash \bigcup_{i=1}^{N_{2k-2}} G_{1i}$, then

$$\begin{aligned}\frac{1}{|I|}\int_I f_1(t)\,dt &= \frac{1}{|I|}\int_{I\cap H_{1k}} f_1(t)\,dt + \frac{1}{|I|}\int_{I\backslash H_{1k}} f_1(t)\,dt\\ &= \frac{1}{|I|}|I\cap H_{1k}| + \frac{1}{|I|}\int_{I\backslash H_{1k}} f_1(t)\,dt\\ &\geq (1-2^{-2k}) - \frac{2}{|I|}|I\backslash H_{1k}| \geq (1-2^{-2k}) - 2^{-2k+1} \geq 1-2^{-2k+2}.\end{aligned}$$

Similarly, if I is any component of $I_0\backslash \bigcup_{i=1}^{N_{2k-1}} G_{1i}$, then

$$\begin{aligned}\frac{1}{|I|}\int_I f_1(t)\,dt &= \frac{1}{|I|}\int_{I\cap H^*_{1k}} f_1(t)\,dt + \frac{1}{|I|}\int_{I\backslash H^*_{1k}} f_1(t)\,dt\\ &\leq -1+2^{-2k+1}.\end{aligned}$$

In general, suppose that disjoint open sets H_n and H_n^* have been defined with $H_n \cup H_n^* = I_0\backslash \bigcup_{i=1}^n F_i$, and proceed to define H_{n+1} and H_{n+1}^* as follows. First, enumerate the component intervals of $I_0\backslash \bigcup_{i=1}^n F_i$ as $\{G_{ni}\colon i=1,2,\ldots\}$. Since F_{n+1} is disjoint from the previous F_i's, $F_{n+1} \subseteq \bigcup_{i=1}^\infty G_{ni}$. Consider one fixed component G_{ni} and let $\{J_{nij}\colon j=1,2,\ldots\}$ denote the component intervals of $G_{ni}\backslash F_{n+1}$. There is an N_1^i such that $|I\cap \bigcup_{j=1}^{N_1^i} J_{nij}| \geq 3|I|/4$ for any interval $I\subseteq G_{ni}$ such that $|I| \geq |G_{ni}|/(n+1)$. If N_k^i has been defined, let N_{k+1}^i be so large that if I is any component of $G_{ni}\backslash \bigcup_{j=1}^{N_k^i} J_{nij}$, then

$$\left| I\cap \bigcup_{j=N_k^i+1}^{N_{k+1}^i} J_{nij}\right| \geq (1-2^{-k-2})|I|.$$

For notational purposes let $N_0^i = 0$. For each $k=1,2,\ldots$, let $H_{(n+1)ik} = \bigcup_{j=N_{2k-2}^i+1}^{N_{2k-1}^i} J_{nij}$ and $H^*_{(n+1)ik} = \bigcup_{j=N_{2k-1}^i+1}^{N_{2k}^i} J_{nij}$. Next, let

$$H_{n+1} = \bigcup_{i=1}^\infty \bigcup_{k=1}^\infty H_{(n+1)ik} \quad\text{and}\quad H^*_{n+1} = \bigcup_{i=1}^\infty \bigcup_{k=1}^\infty H^*_{(n+1)ik}.$$

Then H_{n+1} and H^*_{n+1} are disjoint open sets and $H_{n+1}\cup H^*_{n+1} = I_0\backslash \bigcup_{i=1}^{n+1} F_i$.

For each j let $J_{nij} = (a_{ij}, b_{ij})$ and define a function $f_{n+1}\colon I_0 \to \mathbf{R}$ by

$$f_{n+1}(x) = \sum_{i=1}^\infty\sum_{k=1}^\infty\sum_{j=1}^\infty \left[\chi_{H_{(n+1)ik}}(x) - \chi_{H^*_{(n+1)ik}}(x)\right]\cdot \alpha\left[\frac{x-a_{ij}}{b_{ij}-a_{ij}}\right],$$

where α is as previously defined.

On each fixed G_{ni} the function f_{n+1} is either continuous or has the same properties as previously described for f_1 except that G_{ni} and F_{n+1} play the roles previously assumed by I_0 and F_1, respectively. In particular, the restriction of f_{n+1}

to G_{ni} is a Baire one Darboux function. Now, if a is any real number, then the associated set

$$\{f_{n+1} < a\} = \left[\{f_{n+1} < a\} \cap \bigcup_{m=1}^{n} F_m\right] \cup \left[\{f_{n+1} < a\} \cap \bigcup_{i=1}^{\infty} G_{ni}\right].$$

Clearly, $\{f_{n+1} < a\} \cap \bigcup_{m=1}^{n} F_m$ is either empty or equals $\bigcup_{m=1}^{n} F_m$, depending upon whether $a \leq 0$, or not. This observation, coupled with the fact that f_{n+1} restricted to each G_{ni} is a Baire one Darboux function, allows us to see that each $\{f_{n+1} < a\}$ is an $\mathcal{F}_\sigma$ set which is bilaterally dense in itself. Since this is also true for associated sets of the form $\{f_{n+1} > a\}$, it follows that f_{n+1} is a Baire one Darboux function (e.g., see Theorem 1.1, part (8), p. 9, in [**1**]).

Furthermore, note that if I is any component of $G_{ni} \setminus \bigcup_{j=1}^{N_{2k-2}} J_{nij}$, then

$$\begin{aligned}
\frac{1}{|I|}\int_I f_{n+1}(t)\,dt &= \frac{1}{|I|}\int_{I\cap H_{(n+1)ik}} f_{n+1}(t)\,dt + \frac{1}{|I|}\int_{H_{(n+1)ik}\setminus I} f_{n+1}(t)\,dt \\
&= \frac{1}{|I|}|I \cap H_{(n+1)ik}| + \frac{1}{|I|}\int_{H_{(n+1)ik}\setminus I} f_{n+1}(t)\,dt \\
&\geq (1-2^{-2k}) - \frac{2}{|I|}|H_{(n+1)k}\setminus I| \geq (1-2^{-2k}) - 2^{-2k+1} \\
&\geq 1 - 2^{-2k+2}.
\end{aligned}$$

Similarly, if I is any component of $G_{ni} \setminus \bigcup_{i=1}^{N_{2k-1}} J_{nij}$, then

$$\begin{aligned}
\frac{1}{|I|}\int_I f_{n+1}(t)\,dt &= \frac{1}{|I|}\int_{I\cap H^*_{(n+1)ik}} f_{n+1}(t)\,dt + \frac{1}{|I|}\int_{H^*_{(n+1)ik}\setminus I} f_{n+1}(t)\,dt \\
&\leq -1 + 2^{-2k+1}.
\end{aligned}$$

We now define our function $f\colon I_0 \to \mathbf{R}$ as

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{4^n} f_n(x).$$

Being a uniform limit of Baire one functions, f is clearly a Baire one function itself. Next, recall that the sum of two Baire one Darboux functions whose sets of points of discontinuity are disjoint is Darboux. (One easy way to see this is to apply W. H. Young's [**7**] criterion for a Baire one function h to be Darboux; namely, that h is Darboux if and only if for each x there exist sequences $x_n \uparrow x$ and $y_n \downarrow x$ such that $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} h(y_n) = h(x)$. This result may be found as part (2) of Theorem 1.1, p. 9, in [**1**].) Since for each N the sets of points of discontinuity of $f_1, f_2, \ldots, f_N$ are pairwise disjoint, it follows that the function $\sum_{n=1}^{N}(1/4^n)f_n$ is a Darboux function. Consequently, the uniform limit function f is both Baire one and Darboux (see Theorem 3.4, p. 15 in [**1**]).

We shall now verify assertion (i) of the theorem statement for this function f. To this end, let $x \in E$. There is a unique N such that $x \in F_N$ and, hence,

$$f(x) = \sum_{n=1}^{N-1} \frac{1}{4^n} f_n(x).$$

Furthermore, since the function $\sum_{n=1}^{N-1}(1/4^n)f_n$ is continuous at x, there is a neighborhood H of x such that if I is any subinterval of H containing x, then

$$\left|\frac{1}{|I|}\int_I \sum_{n=1}^{N-1}\frac{1}{4^n}f_n(t)\,dt - f(x)\right| < 6^{-1}4^{-N}.$$

Since $x \notin \bigcup_{n=1}^{N-1} F_n$, there is a component $G_{(N-1)i}$ of $I_0 \setminus \bigcup_{n=1}^{N-1} F_n$ which contains x. Furthermore, for each k there is some component, say I_k, of

$$G_{(N-1)i} \setminus \bigcup_{j=1}^{N_k^i} J_{(N-1)ij}$$

which contains x. Choose K so large that for all $k \geq K$ we have $I_k \subseteq H$.

Now suppose that $k \geq K$ and k is even. Then

$$\begin{aligned}\frac{1}{|I_k|}\int_{I_k} f(t)\,dt &= \frac{1}{|I_k|}\left(\int_{I_k}\sum_{n=1}^{N-1}\frac{1}{4^n}f_n(t)\,dt + \int_{I_k}\sum_{n=N}^{\infty}\frac{1}{4^n}f_n(t)\,dt\right)\\ &> f(x) - 6^{-1}4^{-N} + \frac{1}{|I_k|}\left(\int_{I_k}\frac{1}{4^N}f_N(t)\,dt + \int_{I_k}\sum_{n=N+1}^{\infty}\frac{1}{4^n}f_n(t)\,dt\right)\\ &> f(x) - 6^{-1}4^{-N} + (1-2^{-k})4^{-N} - \sum_{n=N+1}^{\infty}\frac{2}{4^n}\\ &= f(x) + (6^{-1} - 2^{-k})4^{-N}.\end{aligned}$$

Similarly, if $k \geq K$ is odd, then

$$\frac{1}{|I_k|}\int_{I_k} f(t)\,dt < f(x) - (6^{-1} - 2^{-k})4^{-N}.$$

Consequently, assertion (i) holds.

It is a much easier matter to verify assertion (ii). Indeed, if $x \notin E$, then for each n, x belongs to either H_n or H_n^* and, consequently, f_n is continuous at x. Thus f, being a uniform limit of functions, each of which is continuous at x, is continuous at x, and the proof is complete for the case where each F_n is perfect.

We turn now to the general case. Here each F_n can be expressed as the disjoint union $F_n = P_n \cup Q_n$ where P_n is perfect (or empty) and Q_n is countable. Let $E_1 = \bigcup_{n=1}^{\infty} P_n$ and $E_2 = \bigcup_{n=1}^{\infty} Q_n$. Let g be the function constructed as in the previous case with the nonempty P_n's playing the role of the F_n's.

The set E_2 is countable and we enumerate it as $\{x_n : n = 1, 2, \ldots\}$. Let J be the union of a sequence of pairwise disjoint intervals I_j, $j = 1, 2, \ldots$, in $(0, \infty)$ such that I_{j+1} lies to the left of I_j for each j, and such that J has upper density one from the right and lower density zero from the right at 0. Let $I_j = (c_j, d_j)$ for each j. Define a function $\beta\colon \mathbf{R} \to \mathbf{R}$ by

$$\beta(x) = \sum_{j=1}^{\infty}\alpha\left[\frac{x - c_j}{d_j - c_j}\right].$$

Then β is clearly a Baire class one Darboux function. In fact, it is continuous everywhere except at $x = 0$ and it assumes every value between 0 and 2 infinitely often in each right neighborhood of 0.

Let h_1 be a function defined on I_0 by $h_1(x) = \beta(x - x_1)$. In general, if $h_1, h_2, \ldots, h_n$ have been defined, then define h_{n+1} as follows. Let J_{n+1} be a right neighborhood of x_{n+1} which contains none of the points $x_1, x_2, \ldots, x_n$ and whose length does not belong to J. Then for each $x \in I_0$, let

$$h_{n+1}(x) = \beta(x - x_{n+1})\chi_{J_{n+1}}(x).$$

Each h_n is a Baire one Darboux function, being discontinuous only at x_n. Furthermore,

$$\limsup_{I \to x_n} \frac{1}{|I|} \int_I h_n(t)\,dt = 1 \quad \text{and} \quad \liminf_{I \to x_n} \frac{1}{|I|} \int_I h_n(t)\,dt = 0.$$

We then define $h\colon I_0 \to \mathbf{R}$ as $h = \sum_{n=1}^{\infty}(1/2^n)h_n$. Using the same reasoning as in the previous case, we see that h is a Baire one Darboux function. It is easy to see that for each $x \in E_2$, $\lim_{I \to x}(1/|I|)\int_I h(t)\,dt$ does not exist and that h is continuous at each point in the complement of E_2.

Finally, let $f = g + h$. Since E_1 and E_2 are disjoint, this function is a Baire one Darboux function and clearly satisfies both assertions (i) and (ii).

LEMMA 1. *If $\mathcal{F}$ is a closed set in $\mathcal{B}^1$, then so is $\mathcal{N} \cap \mathcal{F}$.*

PROOF. Suppose that $f_n \in \mathcal{N} \cap \mathcal{F}$, $n = 1, 2, \ldots$, and that $\{f_n\}$ converges (uniformly, of course) to f. We shall show that $N(f) \subseteq \bigcup_{n=1}^{\infty} N(f_n)$, yielding the σ-porosity of $N(f)$. To this end, suppose that $x_0 \notin \bigcup_{n=1}^{\infty} N(f_n)$ and let $\varepsilon > 0$. There is an m such that $|f_m(x) - f(x)| < \varepsilon/3$ for every x in $[0, 1]$. For this m there is a $\delta > 0$ such that if h_1 and h_2 are any nonzero numbers of absolute value less than δ, we have

$$\left| \frac{1}{h_1} \int_0^{h_1} f_m(x+t)\,dt - \frac{1}{h_2} \int_0^{h_2} f_m(x+t)\,dt \right| < \varepsilon/3$$

and, consequently,

$$\begin{aligned}
&\left| \frac{1}{h_1} \int_0^{h_1} f(x_0+t)\,dt - \frac{1}{h_2} \int_0^{h_2} f(x_0+t)\,dt \right| \\
&\quad \le \left| \frac{1}{h_1} \int_0^{h_1} f(x_0+t) - f_m(x_0+t)\,dt \right| \\
&\qquad + \left| \frac{1}{h_1} \int_0^{h_1} f_m(x+t)\,dt - \frac{1}{h_2} \int_0^{h_2} f_m(x+t)\,dt \right| \\
&\qquad + \left| \frac{1}{h_2} \int_0^{h_2} f_m(x_0+t) - f(x_0+t)\,dt \right| \\
&\quad \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Therefore, $x_0 \notin N(f)$, and the proof is complete.

THEOREM 2. *Let $\mathcal{F}$ be any of the subsets, $\mathcal{B}^1, \mathcal{DB}^1, b\mathcal{B}^1$, or $b\mathcal{DB}^1$, of $\mathcal{B}^1$. Then $\mathcal{N} \cap \mathcal{F}$ is a closed nowhere dense subset of $\mathcal{F}$.*

PROOF. Let $\mathcal{F}$ be any of the four classes mentioned in the theorem statement. From the lemma we conclude that $\mathcal{N} \cap \mathcal{F}$ is closed. Now, suppose that $g \in \mathcal{N} \cap \mathcal{F}$ and

let $\varepsilon > 0$ be given. Let $C(g)$ denote that dense $\mathcal{G}_\delta$ set consisting of all continuity points of g. According to Theorem 2 in [**6**], there is a perfect, non-σ-porous set E of measure zero contained in $C(g)$. Now, let f be the function constructed in Theorem 1 of the present paper using this set E. Let $h(x) = g(x) + \varepsilon f(x)$. Since the sets of discontinuity of g and f are disjoint, h has the Darboux property if g does. Consequently, whichever of the four cases is under consideration, it is valid to assert that $h \in \mathcal{F}$. It is now an easy matter to verify that $N(h) \supseteq N(f) \backslash N(g) = E \backslash N(g)$, implying that $N(h)$ is non-σ-porous. Hence, $h \in \mathcal{F} \backslash \mathcal{N}$, from which it follows that $\mathcal{N} \cap \mathcal{F}$ is nowhere dense in $\mathcal{F}$.

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