PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 3, November 1986

A SOFIC SYSTEM WITH INFINITELY MANY MINIMAL COVERS

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ABSTRACT. We give an example of a sofic system S together with an infinite collection of nonconjugate subshift of finite type covers of S such that none of these covers factors through another subshift-of-finite-type cover to which it is not topologically conjugate.

1. Introduction. We answer a question of Boyle, Kitchens, and Marcus [**BKM**] concerning covers of sofic systems. A *factor map* between two symbolic systems is an onto, continuous shift-commuting map. A system which is a factor of a subshift of finite type (SFT) is called a *sofic system*. If π is a factor map from a SFT Σ to a sofic system S we call (Σ, π)—or, for brevity, π—a *cover* of S.

If (Σ, π) and (Σ', π') are covers of S we say π *factors through* π', or π' *intercepts* π, if there is a factor map $\theta\colon \Sigma \to \Sigma'$ for which

$$\begin{array}{ccc} \Sigma & \xrightarrow{\theta} & \Sigma' \\ & \pi\searrow \quad \swarrow\pi' & \\ & S & \end{array}$$

commutes. If θ is invertible (i.e., a topological conjugacy) we say π and π' are *conjugate over* S. Note that this is stronger than saying π and π' are *topologically conjugate* covers [**BKM**], which would require invertible θ and ϕ with

$$\begin{array}{ccc} \Sigma & \xrightarrow{\theta} & \Sigma' \\ \pi\downarrow & & \downarrow\pi' \\ S & \xrightarrow{\phi} & S \end{array}$$

commuting. Departing from [**BKM**] we call a cover π *minimal* if any π' which intercepts π must be conjugate to π over S.

For any sofic S, a well-known construction (see [**F**], [**K**]) gives the *minimal past cover* (Σ_S^-, π_S^-) and *minimal future cover* (Σ_S^+, π_S^+) of S. These are indeed minimal in the sense defined above. S is of *almost finite type* (AFT) if it has a cover π which is one-to-one on a nontrivial open set. In [**BKM**] it was shown that S is AFT if and only if it has a unique minimal cover (up to conjugacy over S); this minimal cover is necessarily conjugate to the minimal past and minimal future covers, and it intercepts every cover of S. If S is not AFT, the minimal past and future covers are not conjugate over S.

Boyle, Kitchens, and Marcus ask the following question: If S is not AFT, is there a finite collection $\{\pi_1, \ldots, \pi_n\}$ of covers such that every cover of S factors

Received by the editors July 22, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54H20, 28D20, 05C20.
Work partly supported by a grant from the University of South Alabama Research Committee.

through some π_i? We answer this question in the negative by exhibiting an infinite collection of minimal covers of a sofic S of which no two are conjugate over S.

2. Example. We use the sofic S described in [**BKM**]. S is a system on the symbols 1, 2, 3, 4, and 5 obtained from the SFT Σ, given by the graph in Figure 1, by a one-block map π which identifies the states 3^{a}, 3^{b}, and 3^{c}. The cover π factors through the minimal past cover via a one-block map identifying states 3^{b} and 3^{c}; similarly, identifying 3^{a} and 3^{b} produces the minimal future cover (Figure 2). (Σ is simply the fibered product of Σ_S^- and Σ_S^+.)

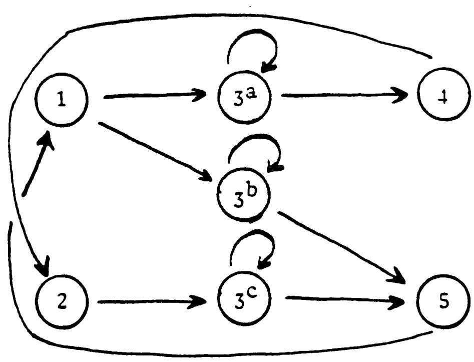

FIGURE 1

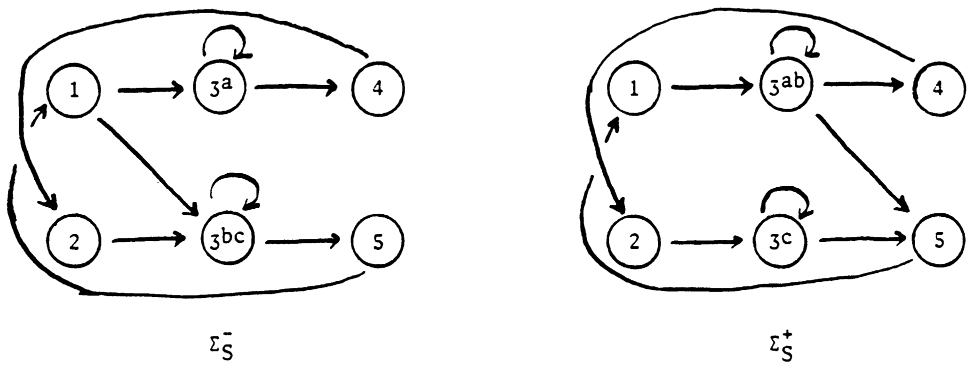

FIGURE 2

For each integer $n > 1$ we first construct a cover (Γ_n', π_n') of S. Γ_n' is given by the graph in Figure 3. The unlabeled states are arranged in n-cycles, and π_n' is the one-block map taking all these states to 3. In effect we have separated out a "zero mod n" component from each fixed point of Σ. Each π_n' factors through π in an obvious fashion.

Finally, we obtain a cover (Γ_n, π_n) by identifying corresponding states in the third and fifth n-cycles from the top in our diagram of Γ_n', and likewise identifying corresponding states in the second and fourth n-cycles (Figure 4). On the "zero mod n" component we have mimicked the map which produces the minimal past cover from Σ, and on the remainder, that which gives the minimal future cover.

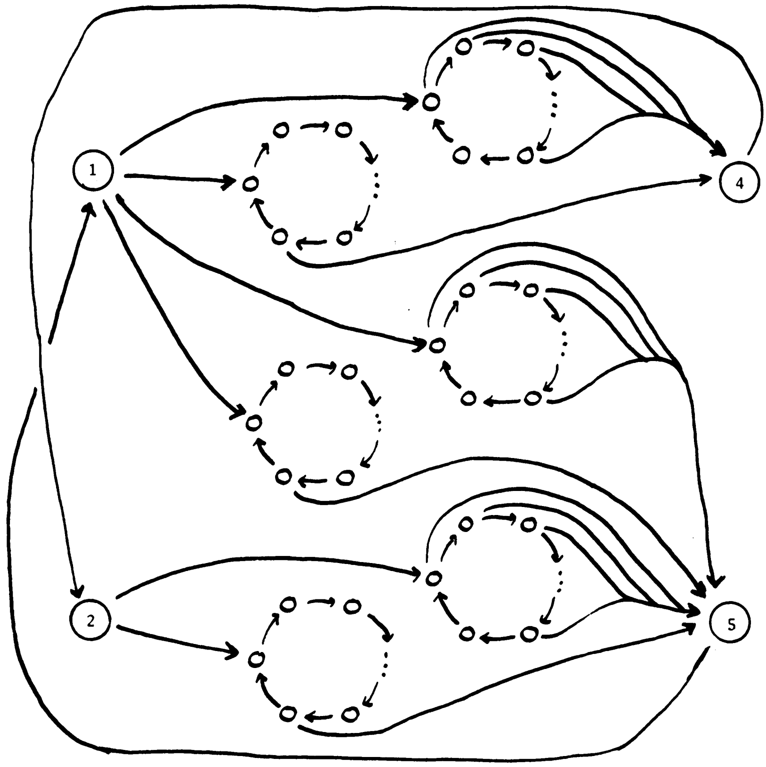

FIGURE 3

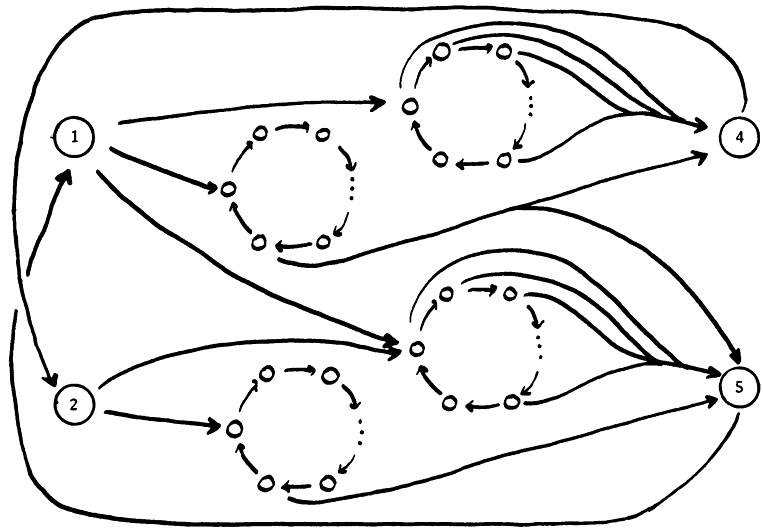

FIGURE 4

For $m \neq n$, π_m and π_n are not topologically conjugate since they have periodic points of different periods. If we had $\pi_n = \Gamma \circ \theta$ with θ a noninvertible map taking Γ_n to another SFT, θ would have to identify at least one pair of periodic points; we leave it to the reader to see that no such identification is possible.

References

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